Can You Prove the Nine-Point Theorem?

Puzzle Corner
by
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Well, Alice and I are settled into our apartment at Rockefeller University. It was sad to have to leave Boston. We rented a truck and did the move ourselves. Unfortunately we picked the hottest day of the year (98°) to load the truck, and the next day we unloaded it in New York in temperate 95°-weather. I've never drunk so much water in my life.

It's-a-small-world department: a familiar face in the Rockefeller University mail room turned out to belong to Sandy Hoffman (née Lazarowitz), who went through my high school and M.I.T. three years behind me. As you can probably guess, she is now a graduate student at "the Rock"—and would you believe she is our next-door neighbor!

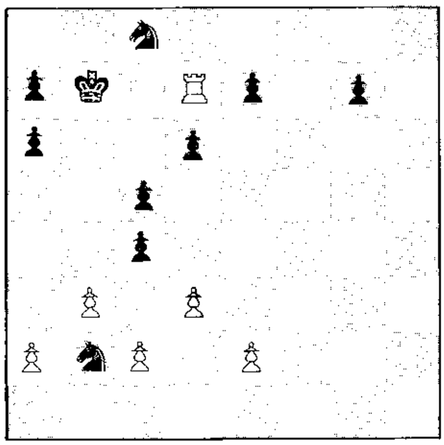
Starting a new volume, here's how Puzzle Corner works: every month we publish five problems and several "speed problems," selected from those suggested by readers. The first selection each month will be either a bridge or a chess problem. We ask readers to send us their solutions to each problem, and three issues later we select for publication one of the answers—if any—to each problem, and we publish the names of other readers submitting correct answers. Answers received too late or additional comments of special interest are published as space permits under "Better Late Than Never." And I cannot respond to readers' answers and queries except through the column itself.

Note: problems for "Speed Department" are in very short supply.

Here goes.

Problems

O/N1 James S. Wasvary has found the board shown at the top of the next column on which a legal chess game has just been played. What piece (black or white) is X?



Zachary Gilstein wants you to prove the nine point theorem:

O/N2 Consider a triangle. Prove that the three midpoints of the sides, the three basepoints of the altitudes, and the three midpoints of the segments joining the vertices to the orthocenter (the common intersection of the three altitudes) all lie on one circle.

The following is from Brian MacDowell

O/N3 Take the digits 1, 9, 7, and 3 and, using any mathematical symbols, construct an expression yielding each number from 1 to ten (each of the four digits must be used exactly once); for example 1^{973} yields one.

O/N4 Winthrop M. Leeds claims that no odd integers A and B satisfy $A^2 + B^2 = C^2$ (of course C would be even). Do you agree?

Here's one from R. Robinson Rowe:

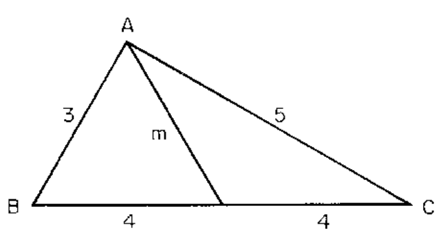
O/N5 For a problem so abSURD that you may work it twice just to be sure, evaluate exactly:

$$X = \sqrt{121 + 15\sqrt{65}} - \sqrt{(11\sqrt{2} - 5\sqrt{5})(11\sqrt{2} - 3\sqrt{13})}$$

Speed Department

SD1 H. W. Hardy wants to know how many combinations of 21 U.S. coins add up to exactly one dollar.

SD2 Norman Brenner wants you to find the length of the median m, below:



Solutions

The following problems were published in the May, 1973, issue of the *Review*.

MAY1 With the hands shown at the top of the next column, South holds a contract for five diamonds. West's lead is ♠4. Do you want to play offense or defense?

North:
♠ K, 7
♥ K, 5, 4, 2
♦ 5, 4, 2
♣ 10, 9, 4, 3

West:
♠ J, 9, 6, 4, 3, 2
♥ Q, 8, 7
♦ 3
♣ K, 6, 2

East:
♠ Q, 10, 8
♥ A, J, 9, 6, 3
♦ K, 8
♣ J, 8, 5

South:
♠ A, 5
♥ 10
♦ A, Q, J, 10, 9, 7, 6
♣ A, Q, 7

The following is from Fred Price: To paraphrase Teddy Roosevelt, the key to the problem is to walk softly and carry a small club—the ♣7 to be exact. South can make his contract as follows: Win the spade lead with the ♠A and lead the ♣Q. If West ducks this, the hand is simple. Cross over to the ♠K, take a winning diamond finesse, and cash out, losing at the end a club and a heart. Suppose West wins the ♣Q with the ♣K—a likely play. His best return is probably the ♥Q, which can cover or not. Assuming the hearts are continued, you trump the second round, cross to dummy's ♠K, and play ♣10. If East covers, win with the ♣A, lead the ♣7 (so carefully preserved) to the ♣9 on the board, take the diamond finesse, and cash out. Should East for some reason duck the ♣10 when it is lead, pitch the ♣7 under it and take the diamond finesse to win again. Any other leads by East/West when they win a trick puts you into the moan sequence that wins the hand.

Also solved by Bob Baird, George Holderness, Jr., Michael Kay, Ron Moore, R. Robinson Rowe, Tom Wagner, and the proposer, Paul Berger.

MAY2 Given that

$$x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}} = 2, \text{ find } x.$$

Recall that in multiple exponentiation, the evaluation starts at the top and proceeds downward. Thus the substitution

$$u = x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}} \text{ yields}$$

$x^u = x^2 = 2$, so that $x = \sqrt{2}$.

Now suppose we try solving the equation

$$x^{x^{x^{\cdot^{\cdot}}}} = 4.$$

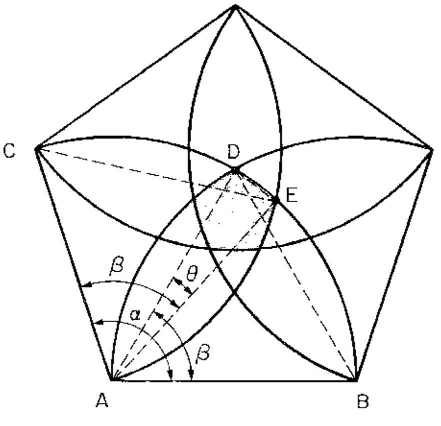
The same reasoning as before leads to the conclusion that $x = \sqrt[4]{4}$.
But the square root of 2 and the fourth root of 4 are precisely the same quantity, both being approximately 1.414. So we are led to the conclusion that 1.414 . . . exponentiated upon itself an infinite number of times, yields both 2 and 4. So 2 = 4. Q.E.D.! What is wrong?

R. Robinson Rowe sent the following elegant solution:

The paradox lies in the set of x for which $x^{x^{x}}$ is convergent. If we let its value be u, then as x increases from 0 to 1.44467, u increases from 0 to e, but if x exceeds 1.44467, the exponentiation diverges and u is infinite. So the premise that u = 4 is an imaginary. The derived relation may be expressed as $x^u = u$. This holds for u from 0 to e, but for greater values of u, its solution is redundant to the primitive relation. This derived relation has an infinite number of pairs, like 2 and 4, which could be used for paradoxes similar to that cleverly presented by Flerser. Another such pair is 2.25 and 3.375. All such pairs are solutions of the interesting equation $x^u = u^x$, which I dubbed the "mutuabola" for a JRM paper. In summary, 1.414 . . . exponentiated upon itself yields 2, but not 4—the latter being a redundant in the derived relation $x^u = u$. The limit of convergence is x = 1.44467 (the e^{th} root of e).

Also solved by Bob Baird, Walter Hill, Neil Judell, Peter Kramer, Albert Mullin, Harry Nelson, John Prussing, and Harry Zaremba.

MAY3 Find the exact area of the shaded space in the pentagon with unit sides:



Harry Zaremba sent in the following:
By hypothesis AD = AE = CE = DB X = 1, triangles ACE and ADB are equilateral, and angle $\alpha = (5\text{-}2)/5 \times 180° = 12°$. Let $\theta = 2\beta - \alpha = 120° - 108° = 12°$. Let c = the length of chord DE = 2 sin $(\theta/2)$. Let A_s = the area of segment DE = ½ $(\theta - \sin(\theta))$. Now A_p, the area of the pentagon within the shaded area, equals 5/4 [c^2 cot (180/5)] = 5/4 [$4a^2 \sin^2(\frac{1}{2}\theta)$ cot (36)] = $5a^2 \sin^2(\frac{1}{2}\theta)$cot (36). Thus the entire shaded area = $A_p - A_s = 5a^2 \sin^2(\frac{1}{2}\theta)$ cot (36) + 5/2 $a^2(\theta - \sin(\theta)$ = 5 sin (6) cot (36) + 5/2 [π/15 − sin (12)], which is approximately 0.0790 square units.

Jordan Backler conjectures that if the pentagon is replaced by an n-gon, the answer is
¼n tan ([(n − 2) (180)]/.2n) − ¼n $\sqrt{3}$ [(6 − n)/6] π (n being at least 4).

Also solved by James Creasy, Neil Judell, Peter Kramer, Charles Landau, R. Robinson Rowe, N. F. Tsang, Don Waldman, Norman Wickstrand, and the proposer, Lee Casperson.

MAY4 Find the quadratic equation with integer coefficients ≤ 10 whose root is the nearest possible approximation to π. (Computer specialists may want to change quadratic to quintic and change 10 to 100.)

John E. Prussing has furnished the following quadratic:
$2x^2 - 5x - 4$. One root is (5 − /57)/4, which is 3.1374578 to eight places. R. Robinson Rowe, without a computer, has found $18x^5 - 27x^4 - 99x^3 - 9x^2 + 100x - 34$, which has a root 3.1415845 to eight places.

Also solved by John Spalding and Harry Zaremba.

MAY5 A tile contractor has laid two floors each composed of 10,000 square pieces —one floor 100 × 100 and the second 80 × 125. What is the total number of squares formed each containing only whole tiles?

The following is from Harry H. Suber:
Suppose that the number of squares of all sizes in a floor which is n × n tiles is N_n. By adding a row of tiles along each side to make the floor (n + 1) × (n + 1) note that the number of 1 × 1 squares is increased by 2n + 1; the number of 2 × 2 squares by (2n − 1); and in general, the number of k × k squares, by (2n − 2k + 1). Thus the number of squares of all sizes added to N_n is

$$(2n + 1) + (2n - 1) \ldots + 1$$

or $(n + 1)^2$ Since $N_1 = 1$ it follows that

$$N_n = 1 + 4 + \ldots + n^2 =$$

$$n[(n + 1)(2n + 1)]/6$$

and $N_{100} = 338{,}350$.

Now, suppose that the number of squares of all sizes in floor n × m, m ≥ n, is M_n. If a row of n tiles is added along the short side to make the floor n × (m + 1) then the number of 1 × 1 squares is increased by n; the number of 2 × 2 square by n − 1; and in general the number of k × k squares by n − k + 1. Thus the number of squares of all sizes is increased by

$$1 + 2 + \ldots + n = n[(n + 1)]/2.$$

Since $M_n = N_n$ when m = n it follows that

$$M_n = n[(n + 1)(2n + 1)]/6 +$$

$$[(m - n)\, n\, (n + 1)]/2.$$

For n = 80, n = 125, M_n = 319,680.

Also solved by Bob Baird, Robert Hisiger, R. Robinson Rowe, L. R. Steffens, and Harry Zaremba.

Better Late Than Never

JA3 The proposor, James C. Wilcox, objects to the given solution. Not being a physicist, I can only print his comments and await adjudication:

The path of the ray of light in the given coordinate system is correct as given by Zaremba and the angle NES is 90°. However, it is not possible to measure this angle with a theodolite since the ray of light passes through point E in the given coordinate system at only a single instant of time. Anyhow, the question asked was, What angles are measured by the observers? The observers are moving with respect to the given coordinate system. It is well known that the state of motion of observers affects such angle measurements (aberration). In order to determine the measured angles correctly, we must find the directions of propagation of the light rays in the reference frames moving with the observers. We shall find the angle measured by observer N. Let the speed of each observer with respect to the given coordinate system be β meters of distance per meter of light-travel time. The equations of motion of observers E and W in the given coordinate system are:

$$E: \quad x = \beta t, \qquad y = 0$$

$$W: \quad x = -\beta t, \quad y = 0$$

We must now find the equations of motion of observers E and W in the reference frame moving with and centered on observer N. The Lorentz transformation from N's reference frame to the given coordinate system is

$$x = x'$$

$$y = y'/\sqrt{1 - \beta^2} + \beta t'/\sqrt{1 - \beta^2}$$

$$t = \beta y'/\sqrt{1 - \beta^2} + t'/\sqrt{1 - \beta^2}$$

Substitution of these expressions for x, y, and t into the preceding equations gives the equations of motion in N's coordinates

$$E: \quad x' = \beta\sqrt{1 - \beta^2}\, t', \quad y' = -\beta t'$$

$$W: \quad x' = -\beta\sqrt{1 - \beta^2}\, t', \quad y' = -\beta t'$$

Thus, at any time, N will find a unit space vector towards E to be

$$(\sqrt{1 - \beta^2}, -1)/\sqrt{2 - \beta^2}$$

and a unit vector away from W to be

$$(\sqrt{1 - \beta^2}, 1)/\sqrt{2 - \beta^2}$$

Since these vectors are not rotating, they must point in the directions of propagation of the light rays in N's frame of reference. Their dot product is the cosine of the measured angle. Thus the angle at N measured by the observer is

$$\alpha = \text{arc cos}\,(-\beta^2/(2 - \beta^2))$$

The other angles must be the same, by symmetry.

Allan J. Gottlieb studied mathematics at M.I.T. (S.B. 1967) and Brandeis (A.M. 1968, Ph.D. 1973) he is now Assistant Professor of Mathematics at York College of C.U.N.Y. Send problems, solutions, and comments to him at the Department of Mathematics, York College, 150-14 Jamaica Ave., Jamaica, N. Y., 11432.