Can You Prove the Nine-Point Theorem?

Well, Alice and I are settled into our apartment at Rockefeller University. It was sad to have to leave Boston. We rented a truck and did the move ourselves. Unfortunately we picked the hottest day of the year (98°) to load the truck, and the next day we unloaded it in New York in temperate 95°-weather. I've never drunk so much water in my life.

It's a small-world after all. As you can probably guess, she is now a graduate student at "the Rock"—and would you believe she is our next-door neighbor?

Starting a new volume, here's how the Puzzle Corner works: every month we publish five problems and several "speed problems," selected from those suggested by readers. The first six problems each month will be either a bridge or a chess problem. We ask readers to send us their solutions to each problem, and three issues later we select for publication one of the answers—if any—to each problem, and we publish the names of all other readers submitting correct answers.

Problems

**O/N1** James S. Wasravy has found the board shown at the top of the next column on which a legal chess game has just been played. What piece (black or white) is X?

Zachary Cileinen wants you to prove the nine point theorem:

**O/N2** Consider a triangle. Prove that the three midpoints of the sides, the three basepoints of the altitudes, and the three midpoints of the segments joining the vertices to the orthocenter (the common intersection of the three altitudes) all lie on one circle.

The following is from Brian MacDowell:

**O/N3** Take the digits 1, 9, 7, and 3 and, using any mathematical symbols, construct an expression yielding each number from 1 to 10 (each of the four digits must be used exactly once); for example 1973 yields one.

**O/N4** Winthrop M. Leeds claims that no odd integers A and B satisfy \( A^2 + B^2 = C^2 \) (of course C would be even). Do you agree?

Here's one from R. Robinson Rowe:

**O/N5** For a problem so absurd that you may work it twice just to be sure, evaluate exactly:

\[
X = \sqrt{121 + 15\sqrt{65}} - \sqrt{(11\sqrt{2} - 5\sqrt{5})(11\sqrt{2} - 3\sqrt{13})}
\]

Speed Department

**SD1** H. W. Hardy wants to know how many combinations of 21 U.S. coins add up to exactly one dollar.

**SD2** Norman Brenner wants you to find the length of the median m, below:

\[
\begin{align*}
\text{A} & \quad 3 \\
\text{B} & \quad m \\
\text{C} & \quad 5
\end{align*}
\]

**Solutions**

The following problems were published in the May, 1973, issue of the Review.

**MAY1** With the hands shown at the top of the next column, South holds a contract for five diamonds. West's lead is A 4. Do you want to play offense or defense?

**MAY2** Given that

\[ X^{xx} = 2, \text{find } x. \]

Recall that in multiple exponentiation, the evaluation starts at the top and proceeds downward. Thus the substitution

\[ u = x^{xx} \]

yields

\[ x^u = x^2 = 2, \text{so that } x = \sqrt{2}. \]

Now suppose we try solving the equation

\[ u = x^{xx} \]

for \( x \) with a calculator. Start by setting \( x = 2 \), and let the calculator take care of the rest. You'll get exactly the same result, provided you didn't forget to set your calculator to the right mode of operation. But remember that you started with a whole number, and the answer is a whole number. The result is quite surprising when you look at it this way.
The same reasoning as before leads to the conclusion that \( x = \sqrt{4} \).

But the square root of 2 and the fourth root of 4 are precisely the same quantity, both being approximately 1.414. So we are led to the conclusion that 1.414 \( \ldots \) exponentiated upon itself an infinite number of times, yields both 2 and 4. So 2 = 4. Q.E.D. What is wrong?

R. Robinson Rowe sent the following elegant solution:

The paradox lies in the set of \( x \) for which \( x^n = u \), which is \( x = \sqrt[n]{u} \). If we let its value be \( u \), then as \( x \) increases from 0 to 1.44467, \( x \) increases from 0 to \( e \), but if \( x \) exceeds 1.44467, the exponentiation diverges and \( u \) is infinite. So the premise that \( u = 4 \) is an imaginary. The derived relation may be expressed as \( x^n = u \). This holds for \( u \) from 0 to \( e \), but for greater values of \( u \), its solution is redundant to the primitive relation. This derived relation has an infinite number of pairs, like 2 and 4, which could be used for paradoxes similar to that cleverly presented by Flecher. Another such path is 2.25 and 3.375. All such pairs are solutions of the interesting equation \( x^n = u^n \), which I dubbed the "mutualisola" for a JRR paper. In summary, 1.414 \( \ldots \) exponentiated upon itself yields 2, but not 4—the latter being redundant in the derived relation \( x^n = u \). The limit of convergence is \( x = 1.44467 \) (the \( e^{th} \) root of \( e \)).


**MAT3** Find the exact area of the shaded space in the pentagon with unit sides:

- The area is \( 0.0750 \) square units.
- Jordan Bickell conjectures that if the pentagon is replaced by an \( n \)-gon, the answer is \( \frac{1}{2} \pi n \tan \left( \frac{(n - 2)(180)}{2n} \right) = \frac{1}{2} \pi \sqrt[2]{(n - 2)/n} \) x (n being at least 4).


**MAT4** Find the quadratic equation with integer coefficients \( \alpha \) \( \leq \) 10 whose root is the nearest possible approximation to \( \pi \). (Computer workers may want to change quadratic to quintic and change 10 to 100.)

John E. Pressing has furnished the following quadratic:

\[ 2x^2 - 5x - 4 = 0 \]

One root is \( (5 - \sqrt{5})/4 \), which is 3.1374578 to eight places. R. Robinson Rowe, without a computer, has found \( 186^8 \approx 27x^3 - 9x^2 - 9x + 100x - 34 \), which has a root 3.14159454 to eight places.

Also solved by John Spalding and Harry Zaremna.

**MAT5** A tile contractor has laid two floors each composed of 10,000 square pieces — one floor 100 by 100 and the second 80 by 125. What is the total number of squares formed containing only whole tiles?

The following is from Harry H. Suber: Suppose that the number of squares of all sizes in a floor which is \( n \times n \) is \( N_n \). By adding a row of tiles along each edge to make this floor \( (n + 1) \times (n + 1) \), we note that the number of \( 1 \times 1 \) squares is increased by \( 2n + 1 \); the number of \( 2 \times 2 \) squares is increased by \( 2n - 1 \); and in general, the number of \( k \times k \) squares, by \( 2n - 2k + 1 \). Thus the number of squares of all sizes added to \( N_n \) is

\[ (2n + 1) + (2n - 1) + \ldots + 1 \]

or \( (n + 1)^2 \) Since \( N_1 = 1 \) it follows that \( N_n = 1 + 4 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6} \)

and \( N_{100} = 358,350 \).

Now, suppose that the number of squares of all sizes in floor \( n \times m \), \( m \geq n \), is \( M_n \). If a row of \( n \) tiles is added along the short side to make the floor \( n \times (m + 1) \) then the number of \( 1 \times 1 \) squares is increased by \( n \); the number of \( 2 \times 2 \) square by \( n - 1 \); and in general the number of \( k \times k \) squares by \( n - k + 1 \). Thus the number of squares of all sizes is increased by

\[ 1 + 2 + \ldots + n = \frac{n(n + 1)}{2} \]

Since \( M_n = N_n \) when \( m = n \) it follows that

\[ M_n = n(n + 1)(2n + 1)/6 + \left( \frac{m(n - n)}{n(n + 1)/2} \right) \]

For \( m = 80, n = 125 \), \( M_{125} = 319,680 \).

Also solved by Bob Baird, Robert Higgin, R. Robinson Rowe, J. R. Steffens, and Harry Zaremna.

**Better Late Than Never**

J3A The proposer, James C. Wilcox, objects to the given solution. Not being a physicist, I can only print his comments and await adjudication:

The path of the ray of light in the given coordinate system is correct as given by Zaremna and the angle NES is \( 90^\circ \). However, it is not possible to measure this angle with a theodolite since the ray of light passes through point \( E \) in the given coordinate system at only a single instant of time. Anyhow, the question asked was, What angles are measured by the observers? The observers are moving with respect to the given coordinate system. It is well known that the state of motion of observers affects such angle measurements (aberration). In order to determine the measured angles correctly, we must find the directions of propagation of the light rays in the reference frames moving with the observers. We shall find the angle measured by observer \( N \). Let the speed of each observer with respect to the given coordinate system be \( \beta \) meters of distance per meter of lighttravel time. The equations of motion of observers \( E \) and \( W \) in the given coordinate system are:

\[ x = x', \quad y = 0 \]
\[ W: \quad x = -x', \quad y = 0 \]

We must now find the equations of motion of observers \( E \) and \( W \) in the reference frame moving with and centered on observer \( N \). The Lorentz transformation from \( N \)'s reference frame to the given coordinate system is:

\[ x = x' + y'(\sqrt{1 - \beta^2} - \beta t), \quad y' = -\beta y' \]
\[ W: \quad x' = -x', \quad y' = \beta x' \]

Thus, at any time, \( N \) will find a unit space vector towards \( E \) to be

\[ \frac{1}{\sqrt{1 - \beta^2}}, \quad 1/\sqrt{2 - \beta^2} \]

and a unit vector away from \( W \) to be

\[ \frac{1 - \beta^2}{\sqrt{2 - \beta^2}}, \quad 1/\sqrt{2 - \beta^2} \]

Since these vectors are not rotating, they must point in the direction of propagation of the light rays in \( N \)'s frame of reference. The dot product is the cosine of the measured angle. Thus the angle at \( N \) measured by the observer is

\[ \alpha = \cos \left(-\beta \angle (2 - \beta)\right) \]

The other angles must be the same, by symmetry.

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