How Four Dogs Meet In a Field, etc.

Puzzle Corner:
Allan J. Gottlieb

Next year Alice and I will be in New York; I'll be in the Mathematics Department of York College, a new campus in the City University of New York system, while Alice will be a graduate student at Rockefeller University.

Here's an indication of how long I've been out of "the city" (old-time provincial New Yawker talking). While crossing the street today I noticed that another pedestrian was nearly run over right in front of me. I gasped. A third pedestrian turned and said, "You must be from out of town. That wasn't even close—he missed him by two inches!" Two inches seemed close to me, but no one else was alarmed.

Help! A critical shortage of "speed" problems!

Even before proceeding to new problems, let me give Harry Nelson a chance for rebuttal on **M1**, originally published in May, 1972, on which we published Hallock G. Cambell's remarks in May, 1973 *(p. 60)*:

"I concur that the laws of chess require 'only a three-time repetition of the same position' but not that it is 'quite independent of the previous moves.'

"The first 17 moves by black and white which I sent in (plus the 18th white move) were needed to establish that the possibility of both black and white castling on either side has not been precluded. Thus with black's 27th move (KR—R2), the situation is changed; now black may no longer castle on the King's side. Similarly at moves 36W, 44B, and 53W. Thus even after move 61W, the same position has not occurred three times, 'with the possible moves of all the men unchanged'."

Problems

A bridge problem from Winslow H. Hartford:

J/A1 South has won a contract of six spades. West's opening lead is ♠3, taken by ♠A. East returns ♠4. How can South make the contract?

North:
♠ 6 5
♥ A K 10 8 2
♦ 10 4
♣ K J 7 3

West:
♠ 3
♥ Q 7 5 4
♦ Q 9 7 3
♣ Q 10 8 5

East:
♠ A 4
♥ J 9 6
♦ J 8 5
♣ A 9 6 4 2

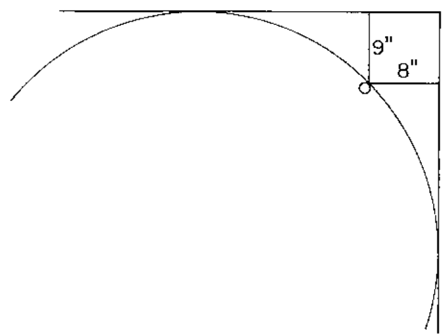
South:
♠ K Q J 10 9 8 7 2
♥ 3
♦ A K 6 2
♣ —

David B. Smith has submitted a problem of a type I find quite enjoyable; he credits it to the late Professor Wroe Alderson of the Wharton School, University of Pennsylvania; Mr. Smith's solution was on Professor Alderson's desk at the time of the latter's death.

J/A2 Each of four dogs, located at the four corners of a square field, simultaneously spots the dog in the corner to his right and runs towards that dog, always pointing directly toward him. All the dogs run at exactly the same speed and thus finally meet in the center of the field. How far did each dog travel?

Our third offering is from Gordon J. McKinnon, Jr.:

J/A3 A circular table is pushed into the corner of a rectangular room. A coin resting on the edge of the table is 9″ from one wall and 8″ from the other. What is the diameter of the table?



Jerry Blum wonders:

J/A4 If you have just been given the dice at a crap table, what are the odds against your winning at least once?

Richard T. Bumby wants you to:

J/A5 Find the greatest common divisor of $a^m - 1$ and $a^n - 1$, where a is a positive integer.

Speed Department

A flagpole problem from H. W. Hardy:

SD1 Three flagpoles of 60 ft., 80 ft., and 100 ft. height are erected at the three corners of a triangular field, 100 ft. on each side. A ladder is placed at a point in the field so that it can be leaned to the exact top of each of the three poles. How long is the ladder, and where is it located?

A baseball puzzle from Norman Brenner:

SD2 The Yankees shut out the Orioles in Yankee Stadium by scoring n runs in the nth inning, for each n. What was the final score?

Solutions

The following are solutions to problems published in the March/April issue of the *Review*—except recall that **M/A1** was modified in June, so we start this month with **M/A2:**

M/A2 If (1) F is continuous from $(0, \infty) \rightarrow (0, \infty)$; and (2) for all $t > 0$, the sequence $F(t), F(2t), F(3t), \ldots \rightarrow 0$; then $F(x) \rightarrow 0$ as $x \rightarrow \infty$ $[x \in (0, \infty)]$.

Jason I. Bitsky, a member of the Department of Mathematics at Princeton, writes, "I offered this problem to a group of hackers at tea. It was quite obvious to everybody that Baire's Category Theorem was called for. The only hitch was how to apply it. The key point (marked after the fashion of chess analyses with a double exclamation point) was brought up by Bob Israel. After that the solution was almost trivial." Here is the complete proof:

Let $\epsilon > 0$ and define for each positive integer n the set $k_n(\epsilon) = \{t > 0: f(nt) \leq \epsilon\}$. It is easy to see that each $k_n(\epsilon)$ is closed in $(0, \infty)$ as follows: The function f_n from $(0, \infty)$ into $(0, \infty)$ defined by f_n: $t \rightarrow f(nt)$ is continuous (because f is continuous), and so $k_n(\epsilon) = f_n((0, \epsilon])$ is the inverse image of a closed set under a continuous function. Now let $A_n(\epsilon) = \bigcap_{m \geq n} k_m(\epsilon) = \{t > 0: f(mt) \leq \epsilon$ for all $m \geq n\}$. Each $A_n(\epsilon)$ is closed (the intersection of closed sets). We claim that $\bigcap_{n=1}^{\infty} A_n(\epsilon) = (0, \infty)$. By hypothesis, $\lim_{n \rightarrow \infty} f(nt) = 0$ for each $t > 0$. This means that for each $t > 0$, there exists a positive integer n(t) such that $f(mt) \leq \epsilon$ whenever $m \geq n(t)$. This is precisely the assertion that $t \in A_{n(t)}(\epsilon)$. Since $(0, \infty)$ is a locally compact Hausdorff space, the Baire Category Theorem implies that some $A_n(\epsilon)$ must have non-empty interior; say $A_N(\epsilon)$ contains an open interval (α, β), $\beta > \alpha$. Choose a positive integer $M \geq N$ so that $^*(M + 1)\alpha < M\beta$ (i.e., $M > \alpha/(\beta - \alpha)$). The inequality * implies that the intervals $(M\alpha, M\beta)$ and $((M + 1)\alpha, (M + 1)\beta)$ overlap. In fact, we have that for all $k \geq 0$ the intervals $((M + k)\alpha, (M + k)\beta)$ and $((M + k + 1)\alpha, (M + k + 1)\beta)$ overlap. Thus $\bigcup_{k>0} ((M + k)\alpha, (M - k)\beta) = (M\alpha, \infty)$. (!!) Now, if $t > M\alpha$, then t lies in some interval $((M + k)\alpha, (M + k)\beta)$, $k \geq 0$. Hence, $f(t) = f(M + k)t/(H + k) \leq \epsilon$, since $t/(M + k) \in (\alpha, \beta) \subset A_N(\epsilon)$ and $M + k \geq N$. Thus, $f(t) \leq \epsilon$ for all $t > M\alpha$, and this says that f(t) tends to 0 as $t \rightarrow \infty$. Remark: the proposition above is perfectly equivalent to the following statement which is interesting in its own right: If U is any unbounded open subset of $(0, \infty)$ there exists a $t > 0$ and a subsequence n_k of the positive integers such that $n_k t \in U$, for all $k \geq 1$. An elementary theorem in analysis asserts that every open set on the real line is a countable union of disjoint open intervals (α_n, β_n). Thus, the proposition above tells us that no matter how we select $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots \rightarrow \infty$, there is always a $t > 0$ such that infinitely many points of the sequence $\{nt\}_{n=1}^{\infty}$ lie in $U = \bigcup_{i=1}^{\infty} (\alpha_i, \beta_i)$, a non-obvious result.

Also solved by Hugh Barrie and the proposer, Mike Rolle.

M/A3 Every day for a week (of seven days) a class of 15 school girls went for a walk. They walked in five rows of three girls each. Each day, each girl had two new "row" mates. How did they do this?

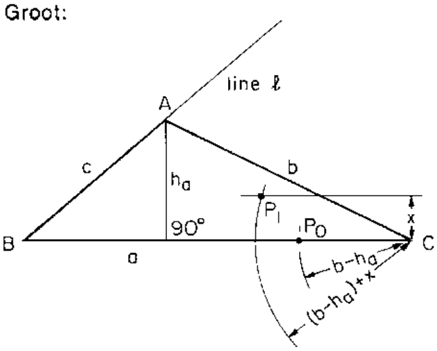
R. Robinson Rowe submitted the following solution and comment: This is the classic problem T. P. Kirkman proposed on page 48 of the *Lady's and Gentleman's Diary* for 1850. It has thousands of solutions, but my favorite is that shown in the box at the top of the next page. The best analysis of the general problem, with the number of girls = G = 3(2d + 1) arranged differently for d days is at pp. 193-223 of W. W. R. Ball's *Mathematical Recreations and Essays* (10th edition, 1937). Henry E. Dudeney in his *Amusements in Mathematics* says, "There are no fewer than 15,567,552,000

Day 1	2	3	4	5	6	7
ABC	ABG	AJM	AEK	AHN	AFO	AIL
DEF	BKN	BEH	CGM	CDJ	BGJ	BDM
GHI	COL	CFI	BOI	BFL	CKH	CEN
JKL	JEI	DKO	DHL	GEO	DNI	FGK
MNO	MHF	GNL	JNF	MKI	MEL	HJO

different solutions," citing the one above.
Also solved by Donald Uhl.

M/A4 Draw a triangle given lengths a and $(b - h_a)$, given angle B, and given that h_a is the perpendicular from A to BC.

The following construction is from Peter Groot:



1. Construct BC = a, angle B giving a line l.
2. Construct from C as a center the distance $(b - h_a)$ toward B on line a. This is point P_0.
3. Construct a line parallel to a at a distance of x above it, and the circle arc of radius $(b - h_a) + x$. The intersection is point P_1.
4. Construct more points P to form a curve; the intersection of this curve with line l will have $x = h_a$ and $AC = (b - h_a) + x = b$. Note that there are two solutions for $b > h_a$, one solution if $b = h_a$, and no solution if $b < h_a$.

Also solved by Mary Lindenberg, R. Robinson Rowe, and David B. Smith.

M/A5 Define $n\Delta = n(n + 1)/2$. When does $n\Delta + 1 = m^2$? Does the following algorithm work for finding the n and m? $n(i) = m(i - 2) * 8 + n(i - 4)$. Given: n and m are required to be integers.

A fine solution from John E. Prussing: Let $T(n) = n(n + 1)/2$ and $p = 2n + 1$. The condition that $T(n) + 1 = m^2$ can then be expressed as $p^2 - 8m^2 = -7$, which is a form of Pell's Equation. A fundamental solution is $p = m = 1$, from which one can ultimately derive the recursion formula $n(k) = 6n(k - 1) - n(k - 2) + 2$. Using $n(0) = -1$, $n(1) = 2$, one generates a sequence of solutions for n. Noting that for each $n - (n + 1)$ is also a solution for the same value of m, since $T[-(n + 1)] = T(n)$, one generates the remaining solutions. The values of m(k) corresponding to n(k) are given by $m(k) = 6m(k - 1) - m(k - 2)$, with $m(0) = 1$, $m(1) = 2$. The solutions are shown in the box at the right.

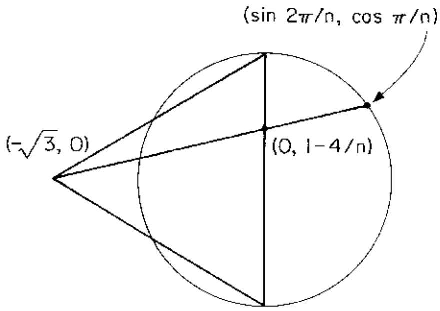
Also solved by John E. Prussing

Better Late Than Never

Concerning **DE1** John Schwartz, the proposer, writes

I take exception to Mr. Cutter's statement in his solution that it is very hard *not* to find a way to make six spades. The point is that the hand was one actually played at the bridge table, and it should be played with the defensive hands unseen. The problem is made trivial if all the hands are visible, in which case it is indeed easy to make the contract.

DE2 Harvey Ploss was going through back issues and found himself particularly interested in Frank G. Smith's construction. He writes: I checked it, and it works for n = 3, n = 4, and n = 6—and perhaps for some other values of n, but it certainly cannot work for all n. Assuming the construction is correct, the diagram below shows what happens. The equation for collinearity is $(\sin 2\pi/n)/\sqrt{3} = (\cos 2\pi/n - 1 + 4/n)/(1 - 4/n)$; it is not satisfied in general. If Mr. Smith's construction were correct he would be able to trisect many angles. Even $20° = 60°/3$ cannot be constructed with ruler and compass because it requires the solution of a cubic.



Richard Lipis, the proposer, feels that the solution given to **DE4** must be wrong, as he knows the numerical solution and the one obtained do not agree. Comments?

Solutions to the following problems have also come from the following readers

DE3 Mary J. Youngquist
DE4 Peter Groot
JN3 Thomas Kauffman
JA1 Roger Sinnott
JA2 Robert Baird and Michael Rolle
JA4 Baron P. de Haulleville, Robert Baird

n	T(n)	T(n) + 1	m:T(n) + 1 = m²
.	.	.	.
.	.	.	.
.	.	.	.
−6	15	16	4
−3	3	4	2
−1	0	1	1
0	0	1	1
2	3	4	2
5	15	16	4
15	120	121	11
32	528	529	23
90	4095	4096	64
189	17955	17956	134
.	.	.	.
.	.	.	.
.	.	.	.

FEB4 Fred Lofgren
FEB5 John Radford

Solutions to Speed Department

SD1 The proposer believes the answer to be a flagpole 105.0206 ft. tall placed 32.0832 ft. from the 100-ft. pole, 68.0392 ft. from the 80-ft. pole, and 86.1936 ft. from the 60-ft. pole.

SD2 I solved this one! The answer: Yankees $\sum_{n=1}^{8} n$, Orioles nothing.

Allan J. Gottlieb studied mathematics at M.I.T. (S.B. 1967) and Brandeis (A.M. 1968, Ph.D. 1973); he is now Assistant Professor of Mathematics at York College of C.U.N.Y. Send problems, solutions, and comments to him at the Department of Mathematics, York College, 150-14 Jamaica Ave., Jamaica, N. Y., 11432.

Book Reviews

Continued from p. 9

A.E.C. has devised for the use of nuclear explosives in civilian application; and the project which is now the center of Commission interest: the breeder reactor. The book makes a substantial attempt to present evenly the opposing arguments on the radiation safety controversies of Ernest J. Sternglass and of John W. Gofman and Arthur R. Tamplin in their book, *Population Control Through Nuclear Pollution.*

H. Peter Metzger, a biochemist who is now a newspaper writer on technical issues, takes broader aim in a more angry book. His purpose is to show how the "Joint Committee and the A.E.C. changed from healthy adversaries into pals: how the Committee was transformed from a critic into an apologist, from an attacker of the A.E.C. into its defender, while the A.E.C. itself was reduced to a fanatically defensive protectionist clique of tenured bureaucrats who have been drawing job security from . . . the Manhattan Project . . . and whose best efforts since then have been divided between widely inappropriate technological adventures and the justification of these past mistakes." His story shows that the weaknesses of the A.E.C. and the supporters are grave indeed and how as a consequence the U.S. nuclear power program is in so much trouble. He invokes most of the known or suspected errors of the A.E.C. in supporting his views, drawn from the U.S. nuclear weapons acquisition and testing programs, the effects of radioactive fallout (the A.E.C.'s "body in the morgue" approach), uranium mine radon, a far wider spectrum of waste storage irregularities than Mr. Lewis documents, and a whole panoply of derelict "atomic gadgets." The latter includes the plutonium-heated "long johns" for deep sea divers produced by the A.E.C. for the Navy, one set of which contains a kilo-