Calculate or Ruminate?

Puzzle Corner
Allan J. Gottlieb

Hi. As you probably know by now, our numbering system has left much to be desired. We'll try to change the system this time, by combining numbers with letters to indicate the month of publication. When you respond to a problem, use both letter *and* number.

I've had the pleasure of a few visitors lately. Harry Nelson came by to show me some of his "puzzle inventions" as well as some conventional puzzles. Last week one of my first M.I.T. roommates, Phil Rosencrantz, dropped by. We were 12 miles apart for the four years after graduation, but we both had to move 3,000 miles before we met.

The response to the chess-bridge alternation has been favorable, so the format will remain.

Remember to send problems and solutions to me at the Department of Mathematics, University of California, Santa Cruz, Calif., 95060; we publish one answer to each problem and acknowledge in print others as received—and we always welcome comments and suggestions.

Problems

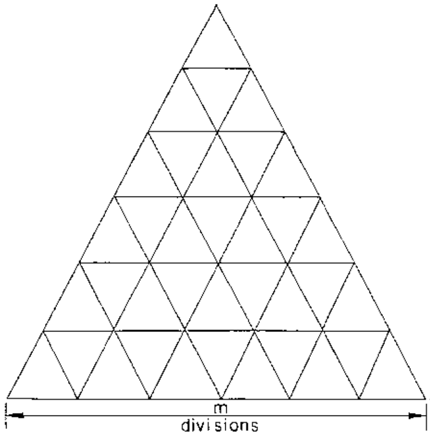
The following chess problem is from Harry Nelson:

M1 In the laws governing the game of chess, provision is made for the claim of a draw under a variety of circumstances. In particular, Law 12 states (in part), "The game is drawn . . . 3. Upon demand by one of the players when the same position appears three times, provided that the same player has the move after each of the three appearances of the same position on the chess board. The position is considered the same if men of the same kind and color occupy the same squares, and the possible moves of all the men are unchanged. . . ." I have found a position reachable by legal play which can occur 21 times without either player being able to demand a draw; can any of your *Technology Review* readers do better?

A sympathetic offering from Stephen Kent:

M2 N soldiers are lined up to be shot. The shooting proceeds as follows: The shooting begins at the left. The first man is skipped, the second man is shot, the third man is skipped, the fourth man is shot, etc., every other man being shot. If a man is not shot, he moves to the end of the line at the right. The last soldier remaining after the shooting is allowed to live. For N = 5, B is shot first, then D, A, and E, and C survives. In general for N soldiers, where should one stand in line if he wishes to live?

A banking puzzle sent in by Harry Zaremba:



M3 The main lobby floor of a new bank building was finished with a mosaic surfacing in which the central feature is a large, contrasting colored triangle patterned from small, equilateral triangles as shown in the illustration above. The floor contractor claimed that it required 10,000 individual units to form the large triangle, and he stated that he could retire comfortably if he had as many dollars as there were triangles of all sizes within the triangular pattern. How many mosaic units m were along each side of the large triangle, and how many triangles of all sizes, including the large one, are to be found within its bounds?

I overheard this one in the mathematics common room at the University of California in Berkeley:

M4 Prove that if 51 integers are chosen from the 100 integers 1, 2, 3, . . . 100, then among the 51 are two a and b such that a evenly divides b.

Winslow H. Hartford offers the following pair of apparently difficult number-theoretic problems:

M5 Find the rational numbers A, B, and C such that

(1) $A^2 - B^2 = 6$, $B^2 - C^2 = 6$, other than the trivial solution $A = 7/2$, $B = 5/2$, and $C = 1/2$.

(2) $A^2 - B^2 = 29$, $B^2 - C^2 = 29$.

Speed Department

Dr. H. B. Levine offers a riddle:

SD1-M "Igor, glorious comrade, now dot you've been elected to de Soviet Applied Problems Society (S.A.P.S.) I'm gung gif you a problem: De sum of de ages from mine sons Alexei, Baliabovitsch and Carlschvanz is precisely nine times Alexei's age. Naxt year de sum vill be six times vot Alexei's age vill be naxt year. And de following year it vill be five times vot Alexei's age vill den be. So, how old are they now?"

"Tovarisch, you're maybe some kind of a nut? It's unpossible to tell!"

"Igor, don't make by me no troubles! Is possible to tell ven you take a serious approach."

Another bank problem, this one from Jack Parsons:

SD2-M In a game between a player and a banker, the latter rolls two dice once. The player then rolls the two dice once. If the player rolls a higher point than the banker, he wins even money; otherwise he loses. What is the player's expectation in this game—that is, over a long series, how much can he expect to win or lose per game?

Solutions

Because of the confusion in numbering, let it be understood that these are solutions to problems which appeared in the January issue of the *Review*:

51 Can the following contract be made against any defense?

North:
♠ 7 3 2
♥ 5 4 2
♦ A 10 9 7 6
♣ 3 2

West:
♠ K 6 4
♥ K J 7 6
♦ K J
♣ K Q J 5

East:
♠ 8
♥ 9 8
♦ 8 5 4 2
♣ 10 9 8 7 6 4

South:
♠ A Q J 10 9 5
♥ A Q 10 3
♦ Q 3
♣ A

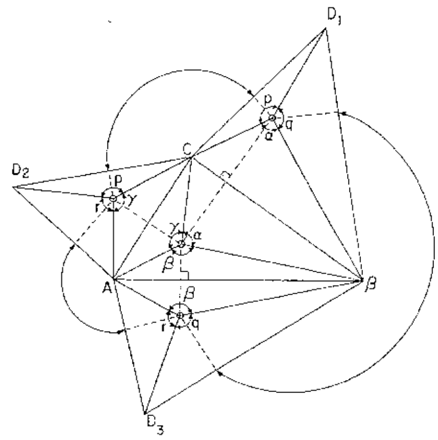
South	West	North	East
1 ♠	Double	2 ♦	Pass
3 ♠	Pass	4 ♠	5 ♣
5 ♠	Double	Pass	Pass
Pass			

This solution is from Charles F. Andrew: The key to this problem is to either gain an entry to dummy through the ♠7 after the diamond suit has been set up or, if West thwarts this, to endplay West. East's cards are immaterial. One diamond trick loss is unavoidable unless West misplays. One trick only can be lost in hearts, or alternatively one in spades if West lets South on the board with the ♠7. For West to keep South off the board requires giving up the ♠K. The play: The best opening lead is ♣5. A diamond lead costs West a trick; a heart or spade lead merely anticipates later plays. South wins the ♣A and leads the ♠Q to remove East's ♠8. West has to hold up the ♠K. South next turns to a diamond to clear the way for setting up North's diamonds; he leads ♦Q. West must cover. North takes the ♦A. North now must lead a low club and trump so that when West is next put into the lead he cannot put South in with a club lead. South now leads the ♦3 and puts West in. West is faced with leading a heart; otherwise he puts North in. It does not matter what he leads; South wins and now leads ♠9. If West ducks he will lose his ♠K to South's ♠A. He must duck, however, to keep South from the board. South then takes West's ♠K. South now leads out his trump and West has to throw his clubs. Finally both South and West have only three hearts each. West is now put on lead with the ♥3 for West's second trick. He is endplayed and must lose the last two hearts to South whatever he leads.

Also solved by 24 other readers—a list simply too long to print here.

52 A sphere is inscribed in a tetrahedron (not necessarily regular). From the four tangency points, lines are drawn to the three adjacent vertices. Prove that the three sets of angles thus formed are equal.

The only response is from Richard Jenney, whose four tangent points are indicated as the points surrounded by small circles in the drawing at the top of the next page.



$$p + q = 360 - \alpha = \beta + \gamma \quad (1)$$
$$p + r = 360 - \gamma = \alpha + \beta \quad (2)$$
$$r + q = 360 - \beta = \alpha + \gamma \quad (3)$$
$$p - q = \beta - \gamma \quad (2) - (3) = (4)$$
$$p - r = \beta - \alpha \quad (1) - (3) = (5)$$
$$r - q = \alpha - \gamma \quad (2) - (1) = (6)$$
$$p = \beta \quad (1) + (4) = (7)$$
$$q = \gamma \quad (3) - (6) = (8)$$
$$r = \alpha; \quad (3) + (6) = (9)$$

α, β, and γ run clockwise in each face. Q.E.D.

53 Given a circle [of radius r and center (h,k)] and a point P [having coordinates (p,q)], find, without using calculus, the coordinates of the point B on the circle having the property that the line segment PB is tangent to the circle.

J. Richard Swenson submitted the following solution:

First, eliminate some unnecessary complexity. Let the coordinate system of the original problem be denoted by (X,Y). Then move the center of the circle to the origin by a simple translation,

$x = X - h, y = Y - k.$

Now change the radius of the circle to 1 by changing the scale of the coordinate system

$U = x/r, V = y/r.$

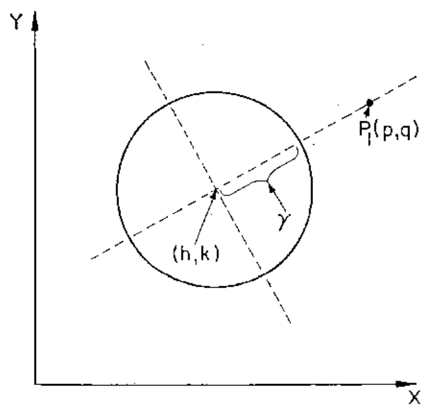
Now rotate the coordinate system so that the point P lies on the horizontal axis with coordinate (s,O):

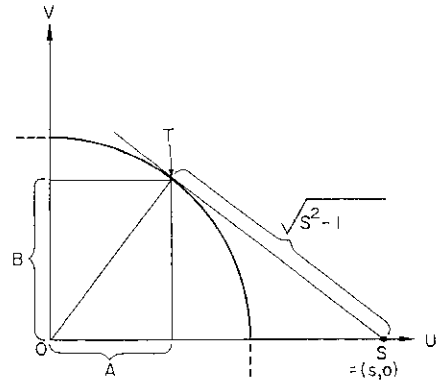
$u = U \cos\theta + V \sin\theta$

$v = V \cos\theta - U \sin\theta$

where $\theta = \arctan (q - k/p - h)$

Geometrically we have





where $s = (\sqrt{(p - h)^2 + (q - k)^2})/r$.

Now given the normalized situation above, it is clear that angle STO is a right angle. Since we know that OT = 1, OS = 1, and $TS = \sqrt{s^2 - 1}$, we can, using similar triangles, deduce that

$A = \cos\phi = 1/s$

$B = \sin\phi = \sqrt{(s^2 - 1)}/s.$

Unwinding the normalization, we find that T has (X,Y) coordinates of

$h + (r^2/[(p - h)^2$
$+ (q - k)^2]) [(q - k)$
$- (\sqrt{(p - h)^2 + q - k)^2 - 1}) (p - h)]$
$k + (\ldots) [(p - h)$
$+ (\sqrt{(p - h)^2 + (q - k)^2 - 1}) (q - k)]$

As a footnote, Mr. Swenson writes that if P is inside the circle, then the above can be reinterpreted as:

Given P inside the circle, determine the tangent points of a single point (S, T, T') such that TPT' is a straight line.

And, he adds, the original problem "is so simple to solve geometrically that I cannot imagine anyone using calculus."

Also solved by Richard Jenney, Burton M. Posthill, R. Robinson Rowe, and Harry Zaremba.

54 All cows eat always at the same rate. Four cows eat three acres of grass in two weeks; three cows eat three acres of grass in three weeks. How long will it take five cows to eat six acres?

Many people got this wrong. A correct solution came from Claude M. Rabache:

A further assumption is that all grass grows always at the same rate g, starting from height H if left alone. Cows eat at rate e but grass reduces at rate (c/a)e where c/a is bovine density in cows per acre. An "eaten" acre is essentially grass that shrinks from H to nothing as cows eat grass faster than it grows (an assumption that will verify itself). Therefore,

$H = t[(c/a)e - g]$. But

$H = 2[(4e/3) - g]$ and $3(e - g)$,

from which

$e = 3g.$

If t_x is the time in question, then

$H = t_x[(5/6)3g - g] = 3(3g - g)$, and

$t_x = 4$ weeks. In general,

$t_x = 6a/(3c - a)$, or

$c/a = 2/t_x + 1/3$

If c/a is such that $(c - a) = 0$, then $t_x = \infty$ and the grass stays at its starting height (at least 1/3 cow per acre). If $(c - a)$ is negative, then the grass will obviously continue to grow past its starting height.

Here are two novel attempts at this problem:

From the Rev. Chris F. Neely:

The problem's hard to calculate,
unless—like cows—you ruminate.
Our problem cows eat four days' food,
but on the fifth cuds must be chewed.
Four hungry beasts graze four days
straight.
Days five and ten they ruminate.
And thus the 14 days have ended
with 48 "grazing days" expended.
Four cows (less one) move to new grass
but cannot eat a bite, alas.
Until day two, then graze for four,
then chew their cud and graze some
more.
Three cows grazed 16 days this round
for 48 grazing days, I found.
Now, any five cows eating tandem
would surely chew their cuds at random.
(Eliminate the ruminant—while four cows
graze,
the fifth one can't).
Six acres grazed, five "average" kine?
Twenty-four days should do just fine!

And one from Julian Pathe:

According to the new permissive math, one pregnant cow eats 1.5 acres in three weeks. At a later date, with all cows using the pill, the original four cows will have increased to five, eating six acres in 3.6 weeks. But the probability of the fifth animal being female is only about 0.5, and we don't know what the bull was doing the night you went to press.

Other attempts and/or solutions came from 22 readers, and we're sorry that space is simply too short to publish the list.

55 Let F_n be the nth Fibonacci number ($F_0 = 0, F_1 = 1, F_i = F_{i-1} + F_{i-2}$). Prove that $F_{n-2}{}^4 \equiv F_{n-1}{}^4 \equiv F_{n+1}{}^4 \equiv F_{n+2}{}^4 \equiv 1 (\text{mod } F_n)$.

Here is Alan LaVergne's answer:

First we observe that since

$F_{n+2} = F_{n+1} = F_{n-1} = - F_{n-2}$
$(\text{mod } F_n)$

it is sufficient to prove that

$F_{n+1}{}^4 = 1 \ (\text{mod } F_n)$.

We will prove that (*)

$F_n{}^2 = F_{n+1} F_{n-1} + (-1)^{n+1}$

This will do the job, since then

$F_{n+1}{}^2 = (-1)^n \ (\text{mod } F_n)$

so that

$F_{n+1}{}^4 = (-1)^{2n} = 1 \ (\text{mod } F_n)$.

For proof of (*): First it is true for n = 2:

$1^2 = 2 \cdot 1 - 1.$

Now if it is true for n, then

$F_{n+1}{}^2 - F_{n+2} F_n = F_{n+1}{}^2$
$- F_{n+1} F_n - F_n{}^2$
$= F_{n+1} (F_{n+1} - F_n) - F_n{}^2 =$
$-(F_n{}^2 - F_{n+1} F_{n-1})$
$= (-1)^{n+2}.$

Also solved by Robert Baird, Winslow H. Hartford, Richard Jenney, John N. Pierce, and R. Robinson Rowe.

Better Late Than Never

Several solutions to problems published in the October/November issue have come in:

41 Mr. and Mrs. Herve Thiriez and Douglas Goodman and **43**

42 Mr. and Mrs. Thiriez

44 Mr. Goodman and Mr. and Mrs. Thiriez

45 Mr. Goodman, H. W. Hardy, and (a beautiful solution) Peter L. Balise

Fred Heutink has supplied solutions to problems **53** and **54** from the December issue.