Puzzle Corner

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Checkmate—In Three Moves

The natural beauty of the Santa Cruz campus (*Mr. Gottlieb is completing his graduate studies at the University of California—Ed.)* could hardly be improved upon. One reason the campus is so lovely is the scarcity of people. There are 4,000 students here and 2,000 acres of woods and meadows occasionally intruded upon by some building. While walking back to the dormitory one evening I met a deer; and after rubbing my eyes I tried to guess the chances of this happening back at M.I.T. The only complaint I have is the proliferation of signs describing appropriate action during an earthquake. I think they're trying to tell me something.

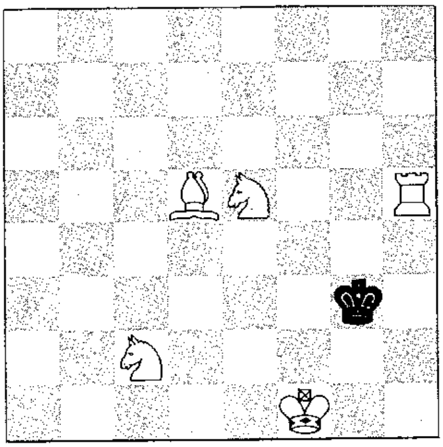
Many people favored chess problems, so they've been included. But there are so many chess and bridge columns that I cannot justify devoting 40 per cent of a Puzzle Corner to these forms of entertainment. Hence we'll alternate chess and bridge, with chess starting out the problems this month.

Remember to send problems, solutions, and comments to me at the Department of Mathematics, University of California, Santa Cruz, Calif., 95060.

Problems

The first chess problem is from Russell A. Hahigian:

51 Given the following, show how white can mate in three moves.

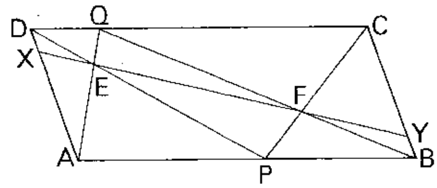


The following is from William H. Bean:

52 A railway car is travelling in a straight line at a constant velocity. A helium-filled balloon is tied to the floor of the car with a piece of string. The car is then decelerated at a constant rate. Relative to the car, what is the initial direction of movement of the balloon? What is the steady-state direction during deceleration (i.e., at what angle is the string to the floor during deceleration)?

Here's a geometry problem from a regular reader, Mrs. Martin S. Lindenberg:

53 Given the parallelogram ABCD with points P and Q located anywhere on AB and CD. Segments BQ, AQ, DP, and CP are drawn; the intersection of AQ and DP is designated E, the intersection of BQ and CP is F. Line EF is drawn intersecting AD at X and intersecting CB at Y. Prove AX = CY.



A number theory challenge from Lawrence H. Smith:

54 If the sum of all the factors of a number equals that same number, that number is called perfect. For example, the first two non-trivial perfect numbers are

$6 = 1 + 2 + 3$

$28 = 1 + 2 + 4 + 7 + 14$

Find a general formula which computes perfect numbers.

Finally, here is a really interesting (and practical) probability problem from Frank Rubin:

55 I own N pairs of socks, all different, which I wash every washday and sort by the following method: all 2N socks are initially in the laundry bag. I withdraw one sock at a time. If its mate is not yet drawn I place it on the bed; if it matches a previously drawn sock I place both in the drawer.

1. What is the probability of the number of socks on the bed becoming 0 between the first and the $(2n - 1)$th draw?
2. What is the expected maximum number of socks on the bed? (that is, the maximum over the 2N-step process)?
3. What is the probability of never having more than M socks on the bed?
4. What is the expected number in the drawer after K selections?
5. What is the expected number of times the pile on the bed will become empty?

Speed Problems

R. E. Crandall asks: Which is greater—1 or $(\sin 1 + \cos \sqrt{2})$?

The following bridge problem was sent to me and signed "courtesy of *The New York Times*": With the hands and bidding shown, what "killing" lead can West make to ensure defeat of the contract against good defense?

North:
♠ Q 2
♥ A K 10 6 5 4
♦ J 8 5 3
♣ 3

West:
♠ K J 10 9 8 7 4
♥ —
♦ A Q 10 4
♣ K 8

East:
♠ 5 3
♥ Q J 9 8 7 2
♦ 9
♣ 9 7 6 4

South:
♠ A 6
♥ 3
♦ K 7 6 2
♣ A Q J 10 5 2

West	North	East	South
1 spade	2 hearts	pass	3 clubs
3 spades	pass	pass	3 n.t.
pass	pass	pass	

Solutions

36 If you are playing South, with clubs distributed as shown:

North
x, x, x
South
Q, J, 9, x, x, x

(the location of ♣ A, ♣ K, ♣ 10, and ♣ x being unknown), how should the suit be played to minimize the loss. You are in your hand with no way of reaching dummy.)

The following analysis is by Michael Kay, making the assumption that the defenders collect all the tricks coming to them: Play a low club first; this wins when the ♣ A or ♣ K is singleton (four times out of 16) and loses when the ♣ 10 is singleton (two times out of 16; the other ten cases are predetermined—the 4 0's and A-K-10-x splits always three tricks and 2-2's only two.

Also solved by Neil Cohen, Winslow H. Hartford, Elmer C. Ingraham, T. D. Landale, John W. Meader, R. Robinson Rowe, John P. Rudy, and Smith D. Turner.

37 A pipe can fill a tank in A hours, and the drain can empty it in B hours. If both are left on at the same time, it takes C hours to fill the tank. Show that there are an infinite number of *integers* A, B, and C which satisfy this problem; and find them.

This one must have been easy; even Joel Shwimer, my old next-door neighbor, solved it: The relation that must be satisfied is $1/A - 1/B = 1/C$, each side showing the fraction of the tank filled in one hour. Any set of integers A, B, C satisfying this relation will solve the problem. Let me consider a subset of all these solutions—namely, those solutions where $B = A + 1$. Now we have $1/A - 1/(A + 1) = 1/A(A + 1) = 1/C$. Therefore, choose any integer A (there are an infinite number of such choices). Let $B = A + 1$ and $C = A(A + 1)$. Both will be integers, and this will satisfy the problem.

Also solved by Robert L. Bishop, Gerald Blum, Neil Cohen, Harold Donnelly, Winslow H. Hartford, Mrs. Martin S. Lindenberg, John E. Prussing, R. Robinson Rowe, Frank Rubin, John P. Rudy, J. Richard Swenson, Paul G. N. de Vegvar, and Harry Zaremba.

38 It is not difficult to prove that $(x^n - x)/n$ is an integer when n is a prime number. (To avoid bickering, let x and n be greater than 1.) Can someone prove, or disprove, that if n is not prime, $(2^n - 2)/n$ cannot be an integer?

The following is from George E. Andrews of the M.I.T. Mathematics Department, who says the assertion of the problem is false. His proof:

When $n = 341 = 11 \cdot 31$, we note that $2^{10} = 1024$ and so $2^{10} - 1 = 1023 = 341 \cdot 3$ or $2^{10} \equiv 1 \pmod{341}$. Hence $2^{341} \equiv (2^{10})^{34} \cdot 2 \equiv 2 \pmod{341}$. If $2^n \equiv 2 \pmod{n}$, then n is known as a pseudo-prime. Sierpinski has pointed out that all of the Fermat numbers Φ_n are pseudo-

primes. This is seen as follows:
$\Phi_n = (2^2)^n + 1$ divides $(2^2)^{n+1} - 1$ divides $[(2^2)^2]^n - 1$ divides $[(2^2)^2]^n + 1 - 2 = 2^{\Phi_n} - 2$.
Φ_1, Φ_2, Φ_3 and Φ_4 are all primes; however, it is not known if any other Fermat numbers are primes. In particular, Euler showed that $(2^2)^5 + 1 = \Phi_5 = 641 \cdot 6700417$. It is known that Φ_n is composite for n = 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 18, 23, 36, 38, 39, 55, 63, 73.

Also solved by Neil Cohen, Harold Donnelly, John W. Langhaar, Ted Leahy, R. Robinson Rowe, Frank Rubin, and Harry Zaremba.

39 Take an arithmetic progression of mn terms and form it into an (m) x (n) matrix by making the first n terms the first row, the next n terms the second row, and so on. What is the rank of this matrix?

Harry Zaremba submitted the following: Let a = first term, d = common difference, and m, n $\neq$ 1. The (m) x (n) matrix will be:

$$\begin{matrix} a & (a+d) & \cdots\cdots & [a+(n-2)d] & [a+(n-1)d] \\ (a+nd) & [a+(n+1)d] & \cdots\cdots & [a+(2n-2)d] & [a+(2n-1)d] \\ (a+2nd) & [a+(2n+1)d] & \cdots\cdots & [a+(3n-2)d] & [a+(3n-1)d] \\ \vdots & \vdots & & \vdots & \vdots \\ [a+(m-2)nd] & \{a+[(m-2)n+1]d\} & \cdots\cdots & \{a+[(m-2)n+(n-2)]d\} & \{a+[(m-2)n+(n-1)]d\} \\ [a+(m-1)nd] & \{a+[(m-1)n+1]d\} & \cdots\cdots & \{a+[(m-1)n+(n-2)]d\} & \{a+[(m-1)n+(n-1)]d\} \end{matrix}$$

If the elements of any column i $\gtrless$ k are subtracted from column k, the elements of column k will all be equal and will be a multiple of the common difference d. Thus, if column 2 is subtracted from column (n − 1), all elements of column (n − 1) will equal (n − 3)d. In any square matrix [A] of third order or greater, the elements in two columns k and j (k $\gtrless$ j) will be multiples of d when the elements of column i are subtracted from them. When the resulting elements of column k are multiplied by a suitable factor and subtracted from column j, all elements of column j will be zero, and therefore the value of det [A] of the square matrix will be zero. Now, since the rank of a matrix is the order of its largest nonvanishing minor, the rank of the matrix above is two—because at least one of its minors of second order is nonvanishing. For example, the value of minor,

$$\begin{vmatrix} a & (a+d) \\ (a+nd) & [a+(n+1)d] \end{vmatrix} = \begin{vmatrix} a & d \\ (a+nd) & d \end{vmatrix} = -nd^2.$$

Also solved by Gerald Blum, Harold Donnelly, John E. Prussing, Carl J. Rosenberg, R. Robinson Rowe, Frank Rubin, and Paul G. N. de Vegvar.

40 Nine men were captured by a strange tribe. All were seated in a straight line. The tribe always served a certain potion to its captives, with every seventh cup containing a deadly poison, and they always served from left to right, continuing from the last victim. They continued serving in this way until all but one prisoner died. They never killed the last man. If you knew this, which position—starting from the left—would you choose?

The last solution is from Paul G. N. de Vegvar:
We write the nine men as follows:
1 2 3 4 5 6 7 8 9.
The first time the potion is served number 7 dies, so we are left with:
1 2 3 4 5 6 8 9.
The second time it is served number 5 dies, so we are left with:
1 2 3 4 6 8 9.
The third time number 4 dies, leaving:
1 2 3 6 8 9.
The fourth time the victim is number 6, leaving:
1 2 3 8 9.
The fifth time, number 9 is gone:
1 2 3 8.
The sixth time it's number 3:
1 2 8.
The seventh time number 8 dies, leaving
1 2.
The eighth time number 1 dies, and number 2 is the survivor. Hence the answer is the second from the left.

Also solved by James W. Avery, Gerald Blum, Harold Donnelly, Winslow H. Hartford, T. D. Landale, Mrs. Martin S. Lindenberg, Michael Kay, John E. Prussing, R. Robinson Rowe, John P. Rudy, Joel Shwimer, and Harry Zaremba.

Better Late Than Never

Apparently the solution to problem **26** in the July/August issue was in error. The following is from James Kempner: Your readers (and you) seem to have been carried away by what appeared to be a successful "Deschapelles Coup" in the bridge problem. The play should show:

Trick	West	North	East	South
1	♣7	♣K	♣A	♣9
2	♦3	♦K	♦Q	♦2
3	?	♣Q	♣2	♥7(!)

If West does not ruff (trivial case), the hand is made, since the Declarer has to lose only two diamonds. If West ruffs, he has three choices: (1) A trump return is taken on the board by the ♠K; the ♣J is then cashed while discarding a diamond from South, leaving only one diamond to lose. (b) A heart return is ruffed by Declarer, the board is entered with a small spade to the ♠K, drawing West's last trump, the ♣J is played . . . (c) If West cashes his diamond and then exits with a diamond, the board ruffs and declarer has no more losers. He only has master trumps in his hand. Furthermore, if West overtakes the ♦Q with his ♦A and shifts to a spade, Declarer is then on the board; he can ruff a small club in his hand with ♠A, draw trump, and enter the board with a diamond to the ♦K and cash his two good clubs, discarding his heart and remaining diamond. If West overtakes the ♦Q with the ♦A and returns a heart to East's ♣A and East returns a club, South ruffs high and is again home free. In conclusion, I see no way to go down at four spades as long as the ♦A is on side giving South two entries to the board.

Other responses to **26** have come from Leonard V. Azaroff, Philip Bell, Harry L. Beohner, Connie Chase, Peter Friedland, David Gross, Leonard Lewis, Peter Lobban, and Frank Westcott.

The solution to problem **30** seems to have ruffled Frank Rubin's feathers a bit. He has written a six-page argument to establish that Mr. Heiberg's solution to his own problem "is about a full of holes as a Swiss cheese." Unfortunately space does not permit publication here, but interested readers may obtain a copy by writing to the Editors of the *Review*.

Other solutions have arrived:
22 Harold Blum
29 Roy G. Sinclair
31 Bowman Cutter