Puzzle Corner
Allan J. Gottlieb

Nine Men Among the Indians

Unfortunately, one portion of the manuscript for this month's column was lost—through no fault of the author's. All the "Better Late Than Never" references, the lists of names from whom solutions have been reveived (the "Also solved by . . ." entries), and several of the credits for problems are gone forever. *The Review* apologizes.

Problems

Someone sent in the following, intended for our mathematical bridge fans:

36 If you are playing South, with clubs distributed as shown:
North
x, x, x
South
Q, J, 9, x, x,x
(the location of ♣ A, ♣K, ♣ 10, and ♣ x being unknown), how should the suit be played to minimize the loss. You are in your hand with no way of reaching dummy.)

Someone else, now unidentified, sent in the following:

37 A pipe can fill a tank in A hours, and the drain can empty it in B hours. If both are left on at the same time, it takes C hours to fill the tank. Show that there are an infinite number of *integers* A, B, and C which satisfy this problem; and find them.

A number theory problem from R. C. Krulish:

38 It is not difficult to prove that $(x^n - x)/n$ is an integer when n is a prime number. (To avoid bickering, let x and n be greater than 1.) Can someone prove, or disprove, that if n is not prime, $(2^n - 2)/n$ cannot be an integer?

Douglas J. Hoylman sends a problem he found while teaching linear algebra:

39 Take an arithmetic progression of mn terms and form it into an mxn matrix by making the first n terms the first row, the next n terms the second row, and so on. What is the rank of this matrix?

Here the last problem for this month, from James R. Bledsoe:

40 Nine men were captured by a strange tribe. All were seated in a straight line. The tribe always served a certain potion to its captives, with every seventh cup containing a deadly poison, and they always served from left to right, continuing from the last victim. They continued serving in this way until all but one prisoner died. They never killed the last man. If you knew this, which position—starting from the left—would you choose?

Speed Problem

Here is a geometrical quickie from Ermanno Signorelli:

SD14 Consider a rectangle with sides a and b, each of arbitrary length, with $a \neq b$. Inscribe five—and only five—triangles in the rectangle, each triangle having two—and only two—sides wholly in common with two other triangles. Identify the positions of the five triangles if the ratio of their areas is 4:5:6:7:8.

Solutions

21 Given the following show how South can complete the contract:

North:
♠ 5
♥ Q 2
♦ A K J 9 8 4 3
♣ J 9 4

West:
♠ 8 3
♥ K J 10 9 8 7 6 3
♦ 7 2
♣ 7

East:
♠ J 7 4
♥ 5
♦ Q 10 5
♣ K Q 10 8 6 3

South:
♠ A K Q 10 9 6 2
♥ A 4
♦ 6
♣ A 5 2

The bidding, North and South being vulnerable:

South	West	North	East
1 spade	4 hearts	5 diamonds	pass
6 spades	pass	pass	pass

West's lead is ♣ 7.

Because "your girl friend likes to see problems solved jointly by husbands and wives," writes Burt Barnow, he and his wife Renee tackled the problem together. Their solution:
South must take the opening club lead in his hand with the ♣ A. He then draws three rounds of trumps with the ♠ A, ♠ K, and ♠ Q (pitching two clubs from dummy). South then leads his ♦ 6 to the dummy's ♦ A. He then leads ♦ K and pitches the ♥ A (!) from his hand. He then leads a third diamond from the dummy and ruffs in his hand. He then leads his ♥ 4 from his hand. If East plays the ♥ K, he is forced to return a heart to dummy's ♥ Q, where there are two good diamonds; if East does not play the ♥ K the dummy's ♥ Q takes the trick. In either case, declarer pitches his two remaining clubs on the two good diamonds in the dummy.
Burt notes that "if my memory is good, you may remember that I was a hall chairman in Baker House at the same time you were." I do; and he was indeed.

22 Show that the series
$1! + 2! + 3! + \ldots + k!$
is asymptotic as $k \to \infty$ to the sum of the last two terms.

The following is from Richard Fistow:
Let

$$s_k = \sum_{i=1}^{k} i!.$$

Then we are asked to show

$$\lim_{k\to\infty} s_k / [(k-1)! + k!] = 1. \quad (1)$$

Since (if $k > 2$) $s_k = s_{k-2} + (k-1)! + k!$, this reduces to

$$\lim_{k\to\infty} s_{k-2}/[(k-1)! + k!] + [(k-1)! + k!]/[(k-1)! + k!] = 1, \quad (2)$$

or

$$\lim_{k\to\infty} s_{k-2}/[(k-1)! + k!] = 0. \quad (3)$$

Now, for $k > 2$ s_{k-2} is the sum of $(k-2)$ terms, the largest of which is $(k-2)!$; so $s_{k-2} \leq (k-2)(k-2)!$.
This allows us to deduce

$$\begin{aligned} 0 &\leq s_{k-2}/[(k-1)! + k!] \\ &\leq (k-2)(k-2)!/[(k-1)! + k!] \\ &= (k-2)/[(k-1) + k(k-1)] \\ &= (k-2)/(k^2-1). \quad (4) \end{aligned}$$

Since the last expression clearly goes to zero as k approaches infinity (an easy proof is to replace k by a continuous variable x and take the limit as x approaches infinity by l'Hopital's Rule), this proves line (3) and completes the solution.

23 From each pound of water passing through a hydraulic turbine we can get more and more energy as we increase the pressure head on the water. It is proposed to place a turbine at the bottom of a tower so high that the energy obtained from each pound of water, when converted to electricity by a generator run by the turbine, will be sufficient to electrolyze that pound of water. The resulting gas mixture, being lighter than air, may rise through an adjoining shaft (wrapped in balloons of infinitesimal weight and infinite stretch, if this idea will help) to the top of the tower, where they may be ignited to reform water, condensed, and returned down the tower. The fact that units in the system are not 100 per cent efficient will not prevent operation, as the tower may be made higher than the theoretical height, producing enough additional power to offset losses. But no perpetual-motion system is economic unless power can be drawn from it. This can be done by making the tower still higher than necessary to electrolyze the water and offset losses; from the lifting effect of the rising gases; from the heat generated by the burning gases; and by use of the superheated steam formed by the combustion to power a turbine. Aside from possible *practical* difficulties (such as the height of the tower):

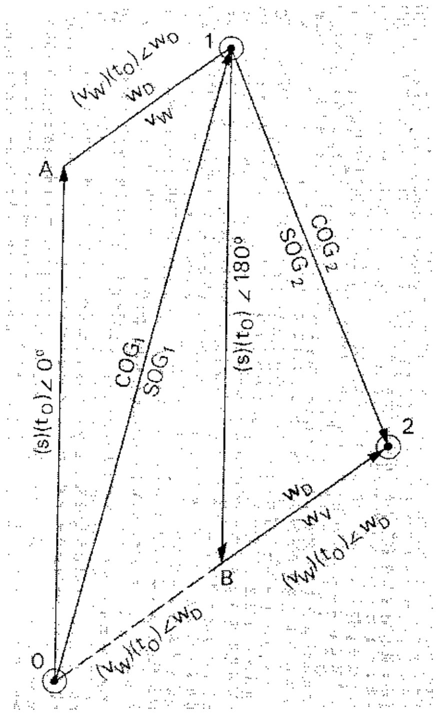
1. Will the system run as described?
2. If so, does it constitute perpetual motion; or, if not, from what source does the energy come?
3. If it would not run, point out any fallacy in the reasoning above.

Stephen S. Flaum set my mind to rest; perpetual motion fails again:
The answer to the perpetual motion problem is that the work needed to expand the hydrogen and oxygen is at least equal to the energy developed by the turbine; the fact that the water is electrolyzed does not significantly change the problem from one in which two pistons are moved away from one another inside a cylinder, creating a vacuum between them. In both cases a pressure is exerted through a volume. The machine will not run. Assuming that the water is of negligible volume, that W pounds of water are being used, that the air is of constant density ρ, and that the air stops at the top of the column whose height is H, then the energy developed by the turbine is given by WH, and the weight of air which must be displaced must equal the weight of the water W. If the volume of air is V,
$V = W$, or $V = W/\rho$.
The work done in expanding the air is the volume through which the air is expanded times the pressure:
Work $= W/\rho \times H\rho = WH$.
This the same as the work developed by the turbine. Actually, of course, my assumptions about the characteristics of the air are not strictly correct, but the same result could be obtained (with considerably more work) by finding

$$\int_{h=0}^{h=H} Pdv.$$

24 An airplane pilot flies a triangular course, flying first due north for a time t_0, then due south for the same time t_0, and finally returning by a straight line to his starting point. The course is triangular because of a wind of unknown direction and velocity V_W. Assuming the pilot has a stop watch and an air speed indicator which shows his speed relative to the air, and he maintains his air speed constant, how much can he determine the direction and velocity of the wind?

I am not an expert on navigation. There is some disagreement concerning the answer to this problem. I mentioned some of this uncertainty in the "lost" manuscript but now can only print this solution, from Dexter R. Wheeler,



which represents one side of the argument; Mr. Wheeler says it "is really a simple navigation problem"; the standard technique, he says, is to consider the effects of the vehicle's engine and the wind separately:

From starting point 0 construct a course line bearing 000° (north) to point A; the length of this vector represents distance and is equal to air speed S multiplied by time t_0. From A draw an arbitrary vector representing wind direction W_D and wind velocity V_W to point 1; the length of this vector represents distance moved due to wind and is equal to velocity V_W multiplied by time t_0, and the vector is oriented with the direction of the wind W_D. The resultant vector $\vec{01}$ represents the actual course and speed made good over the ground. From point 1 construct a new course line bearing 180° (south) to point B; the length of this vector represents distance and is equal to air speed S multiplied by time t_0, and it is parallel to and equal to $\vec{0A}$. From B construct vector $\vec{B2}$ parallel to and equal to vector $\vec{A1}$, since the same wind direction and velocity act upon the plane on course from 1 to B as did from 0 to A. The resultant vector $\vec{12}$ represents the actual course and speed made good over the ground on this day. From point 2, the plane flies back to point 0. The pilot is then heading directly into the wind and so wind direction is determined as plane heading minus 180°. From point 2 to point 0 the plane's speed over the ground is obviously $S - V_W$. The distance to be travelled is $2V_W t_0$, and the time to return t_r is measured by stopwatch. Plugging these values into the general speed-time-distance equation gives $S - V_W = 2V_W t_0/t_r$, and—solving for wind velocity, $V_W = s/(2t_0/t_r) + 1$.

25 Suppose that a football team scores only touchdowns and points-after-touchdown—i.e., that all scores are either 6 or 7 points. Then 12, 13, and 14, for examples, are possible final scores, but 11 and 15 are not. What is the highest unattainable score? In "generalized" football, the only possible scores are a and b, both integers being greater than 1. Under what conditions is there a maximum attainable total score, and what is it?

Generalized football has been put to rest by the following solution from Robert L. Bishop; now if Namath's wrist will heal, next fall should be quite enjoyable:
There is a maximum unattainable score when a and b are relative primes—i.e., when their greatest common divisor is one. If the g.c.d. were two or more, any score not divisible by that number would obviously be unattainable. When a maximum unattainable score does exist, it equals $ab - a - b$; e.g., 29 when $a = 6$ and $b = 7$.
My proof (devised with the casual but vital help of my colleague, R. M. Solow) makes use of the theorem (which can be proved, though I merely assert it here) that there exist two linear combinations of the relative primes a and b that involve only positive integers, with $m < b$, $n < a$, $s < b$, and $t < a$:

$$ma - nb = 1, \quad (1)$$

$$-sa + tb = 1. \quad (2)$$

Let x be the maximum unattainable score. The next higher score, being attainable, may then be expressed in terms of some non-negative integers, h and k:

$$x + 1 = ha + kb. \quad (3)$$

If we subtract first (1) and then (2),

$$x = (h - m)a + (k + n)b,$$

$$x = (h + s)a + (k - t)b.$$

The unattainability of x must now be reflected in the fact that $h - m < 0$ and $k - t < 0$. Moreover, the *largest* values of h and k for which that will be true are $h = m - 1$ and $k = t - 1$.
Therefore, by substitution in (3), we get:

$$x = (m - 1)a + (t - 1)b - 1. \quad (4)$$

Subtracting (2) from (1) gives:

$$(m + s)a = (n + t)b.$$

Furthermore, since $m + s < 2b$ and $n + t < 2a$, and since a and b are relative primes, the only integer solutions are such that $a = n + t$ and $b = m + s$. Now, by substituting $s = b - m$ in (2), we get:

$$-(b - m)a + tb = 1$$

$$ma + tb - 1 = ab$$

$$(m - 1)a + (t - 1)b - 1 = ab - a - b.$$

Therefore, from (4),

$$x = ab - a - b.$$

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