The "Most Impressive" Solution

Yesterday I gave an examination to my calculus class which was supposed to last 1½ hours. After 2½ hours the last unhappy students finally left. Today in class I had to face them personally. As a result, I feel quite guilty about wasting other people's time; so for a change, I'll be brief.

The current backlog is rather large (some problems used this month were submitted last July), so please be patient. I am sincerely flattered by the response this column has received. I must have over 100 letters in front of me now, and it is nice to feel wanted.

Problems

Our bridge problem for this month is from Edwin G. Davis:

31 South has rashly bid six spades:

North:
♠ 2
♥ A Q 10 9 8 7
♦ A K Q
♣ J 3 2

West:
♠ 6 5 4 3
♥ K J
♦ 8 7
♣ K Q 10 9 4

East:
♠ 10 9 8 7
♥ 6 5 4 3 2
♦ 6 5 4 3
♣ —

South:
♠ A K Q J
♥ —
♦ J 10 9 2
♣ A 8 7 6 5

How can South make the contract against a spade lead?

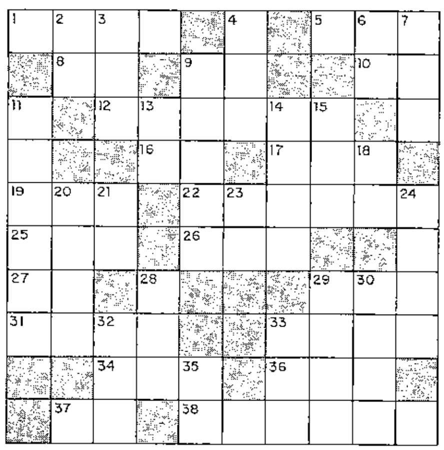
32 Norman L. Apollonio wants you to prove that a regular icosahedron having the same volume as a regular dodecahedron has the same perpendicular distance from the center to a face.

The following game, submitted by John E. Prussing, was discovered by David L. Silverman:

33 Two players play a game in which each player alternately selects dates of the year subject to the restrictions: (1) The first date must be in January, and each subsequent selection must (2) agree with the immediately preceding date in either month or number and (3) be later in the calendar year. The winner is the player who is able to select December 31. Which player has the advantage and what is the winning strategy?

John Mandl sent a copy of this problem to the Editor of the Journal of the Society of Aeronautical Weight Engineers; only two solutions have come in from a readership of 1,000 members. So here's our chance to beat the S.A.W.E. at their own game:

34 The problem is to determine the total quantity of vehicles (both types A and B) purchased (horizontal 5). It is known that the center of gravity (CG) of type A vehicles is identical to the center of gravity of type B vehicles. The customer bought some A vehicles and some B vehicles. All values are positive whole numbers. There were 46 charter members of S.A.W.E. Use the speed of light as 186,284 mi./sec., and 4,633 lb. in. sq. per slug ft. sq. There is one unique solution, requiring only basic mathematics, some logic, and a lot of ingenuity.



Horizontal:
1. Weight of vehicle A.
5. Total quantity of vehicles A and B purchased.
8. Sum of digits of 22 horiz. and 11 vert.
9. Robert's age.
10. Age of the second Fred in Robert's family.
12. Total cost of vehicles A and B purchased.
16. Sum of the ages of the three Freds in Robert's family.
17. CG of forward section of vehicle B.
19. Weight of aft section of vehicle A.
22. Moment from station O of vehicle A.
25. Unit price of vehicle A.
26. Weight of forward section of vehicle A.
27. Weight-reduction ideas considered, but not implemented, for vehicle B.
29. Non-prime number.
31. Next in series 7, 6, 5 . . .
33. Product of the ages of the three Freds.
34. Weight of vehicle B.
36. Unit price of vehicle B.
37. Age of eldest Fred.
38. Moment from station O of vehicle B.

Vertical:
2. Cost-increase weight-reduction ideas implemented in vehicle B.
3. CG of vehicles A and B.
4. CG of aft section of vehicle A.
6. Cost-decrease weight-reduction ideas implemented in vehicle B.
7. CG of aft section of vehicle B.
9. Robert discovered that when his name was arranged as a division problem, there were four numerical solutions:

$$\begin{array}{r} \text{ENNIS} \\ \text{J}\overline{)\text{ROBERT}} \end{array}$$

One quotient is 311 - - .
11. Speed of light (*).
13. Greater than 60.
14. Greater than the square of the age of the eldest Fred.
15. Total weight-reduction ideas considered for vehicle B.
18. LXXVI.
20. Lb. in. sq. per slug ft. sq.
21. No-cost-change weight-reduction ideas implemented in vehicle B.
23. Robert's age. He is 12 years older than the middle Fred.
24. Square of the number of charter members of S.A.W.E.
28. Quantity of vehicles B purchased.
29. Square of atomic number of krypton.
30. Year of election of the 70th U. S. Congress.
32. Weight of forward section of vehicle B.
33. Decimal equivalent of the binary 10,000,011.
35. Prime number.

It seems unlikely to me that anyone but the proposer, Preston Bush, will solve this one, but I've been wrong before:

35 Given the following sequence of numbers:
1,2,1,3,2,1,2,3,4,1,2,1,3,2,1,2,3,5,2,3,4,1,4 3,1,4,5,2,3,1,5,2,1,2,4,6,1,2,1,4,2,1,2,6, . . .
find the method of formation. Of interest—though of little help in solving the problem—is where numbers first appear; for example, 5 first appears at the 18th position:

Number	Position of first appearance
1	1
2	2
3	4
5	18
6	36
7	67
8	94
9	148
10	178
11	301
12	337
13	452
14	606
15	612
16	736
17	986
18	1120

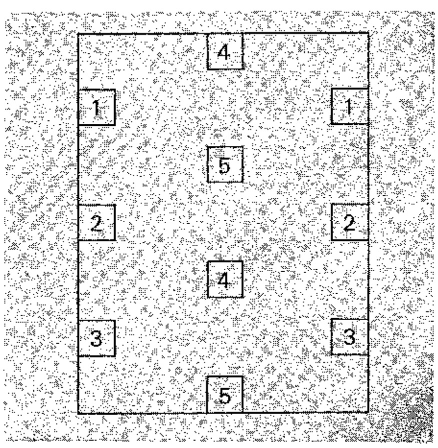
Speed Department

R. Robinson Rowe submits the following, a problem which he used in the 1940's to try to help some of his colleagues in the California Division of Highways prepare for the examination for professional registration, which suddenly became a requirement for advancement. Equations of energy and momentum led into quadratics—hence the problem:

SD9 Two balls approach head-on, Ball A eastbound at 10 knots and Ball B westbound at 15 knots; after colliding, Ball A is westbound at 20 knots and Ball B eastbound at 5 knots. Suppose the approach velocities had been 15 knots for A and 10 knots for B, what would have been their respective retreat velocities?

Here is a beauty from John P. Rudy:

SD10 Connect the pairs of points, 1 with 1, 2 with 2, etc., without crossing lines, going outside boundaries, or going within the eight little squares.



Solutions

16 Given the following hand:

North: ♠ Q x x x; ♥ Q x x; ♦ A K x; ♣ K 9 x

West: ♠ 10 9 x; ♥ A; ♦ x x x; ♣ A Q 10 x x x

East: ♠ J; ♥ K J 10 x x x; ♦ J x x x; ♣ J x

South: ♠ A K x x x; ♥ x x x; ♦ Q x x; ♣ x x

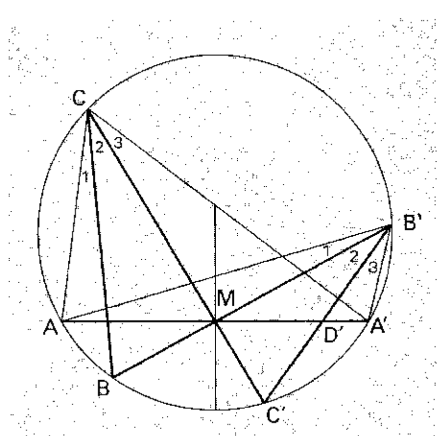
West opens with the ♥A and shifts to a low club. Can you make the contract—four spades?

The following is by Eugene D. Richter: In order to have a chance of making his contract, the declarer initially has no choice but to come up with dummy's ♣K at trick 2, followed by drawing three rounds of trumps. Five more tricks are then safely available in diamonds and trumps, for a total of nine tricks. The tenth trick is established either by a ruff or by setting up dummy's ♥Q. After clearing the diamond suit, a club is led from either hand. If West takes the trick with the ♣A or ♣Q, he must then return the other high club which the declarer permits him to hold by discarding a heart from the closed hand. Still in the lead, West must continue the ♣10 which is ruffed in dummy. This permits the final heart loser to be discarded from the closed hand, leaving it with two good trumps, thus securing the contract. If West plays low on the declarer's club lead, East wins the trick with ♣J. Assuming he has not discarded the 13th diamond, East might choose to lead it, but this would give the declarer a heart discard in the closed hand while ruffing the diamond lead in dummy. Dummy's ♥Q then falls to East's ♥K, and the closed hand claims the remaining two tricks in trumps. East's other option is to lead a heart which sets up dummy's ♥Q for the declarer's tenth trick. The losing tricks are either two clubs and a heart or two hearts and a club, depending on which defender takes the club trick.

Also solved by W. C. Backus, Richard A. Bator, F. Steele Blackall, III, Ed Gershuny, Winslow H. Hartford, Allen J. Huber, Michael A. Kay, John S. Larsma, John W. Meader, Frank S. Model, R. Robinson Rowe, Les Servi, Hugh D. Sims, P. J. Sullivan, and Eric Weitz.

17 Let the chord AA′ perpendicular to a radius of a circle intersect the radius in M. Let any other two chords through M intersect the circle in B,B′ and C,C′. Let BC and B′C′ intersect AA′ in D and D′. Show that MD = MD′.

The following elegant solution is by John T. Rule; he calls it "a beautiful example of the power of the cross ratio theorem":



This is the "butterfly" problem. It is quite difficult to prove by a straightforward Euclidian attack. The proof is, however, quite simple if the cross ratio theorem is employed. Two sheaves of rays C(ABC′A′) and B′(ABC′A) are congruent since the angles are respectively equal, as they subtend equal arcs. Hence transversals cutting these sheaves will yield point rows having equal cross ratios when taken in the same order. Consider the transversal AA′; we may write the equality of cross ratios as follows:
For sheave B, AD′/D′M ÷ A′A/AM =
For sheave C, A′M/DM ÷ A′A/AD.
Remembering that A′M = AM, this equality by cancellation will reduce to AD′/D′M = AD/DM.
But AD′ = A′M − D′M, and
AD = AM − DM, so
(A′M − D′M)/D′M = (AM − DM)/DM, or
(A′M/D′M) − 1 = (AM/DM) − 1,
and since A′M = AM, D′M = DM.

Also solved by W. M. Burgess, John L. Joseph, Mrs. Martin S. Lindenberg, Michael Rolle, R. Robinson Rowe, and F. H. E. Vose.

18 A railroad operates under the following conditions: 1. There is exactly one train per day to take passengers from any given suburban station to any other suburban station; 2. No two trains in the same direction have more than one stop in common; 3. Each train stops at exactly three suburban stations; and 4. More than one train per day in each direction stops at each suburban station. How many trains per day are there in each direction, and how many stations are there on the line?

The following is from Robert W. Baird: We can treat the problem as if trains travel in one direction only, since the number of trains in each direction must be equal. Call the number of stations n and the number of trains in a given direction t. By conditions 1 and 3, each train which stops at a given station must also stop at two of n − 1 other stations, and by condition 2 these trains have only the given station as a common stop. Thus (n − 1)/2 trains stop at each station, making n(n − 1)/2 total stops altogether and requiring n to be odd. By condition 3 again, there must be n(n − 1)/6 total trains. Therefore, there are an infinite number of solutions all satisfying the Diophantine equation: t = n(n − 1)/6 (n odd and n > 3); condition 4 eliminates the solution t = 1, n = 3. Two sample solutions:

If n = 7, t = 7, the trains stop at stations as follows:

Train 1: 1,2,3	Train 5: 2,5,7
Train 2: 1,4,5	Train 6: 3,4,7
Train 3: 1,6,7	Train 7: 3,5,6
Train 4: 2,4,6	

If n = 9, t = 12, the trains stop at stations as follows:

Train 1: 1,2,3	Train 7: 2,6,9
Train 2: 1,4,5	Train 8: 3,4,6
Train 3: 1,6,7	Train 9: 3,5,9
Train 4: 1,8,9	Train 10: 3,7,8
Train 5: 2,4,8	Train 11: 4,7,9
Train 6: 2,5,7	Train 12: 5,6,8

Also solved by Roger Milkman, Michael Rolle, and R. Robinson Rowe.

19 Rationalize the denominator of

$$\frac{1}{\sqrt{2} - \sqrt[3]{3} - \sqrt[5]{5}}$$

or prove it impossible.

R. Robinson Rowe submitted the most impressive solution I have ever received. His *magnum opus* is reprinted below. A gold border would be fitting, but unfortunately my congratulations will have to suffice. Here is the answer:

The problem is to rationalize the denominator of a fraction or prove it impossible. The denominator is the algebraic sum of three surds. In general, it is possible to rationalize the sum of any number of surds by multiplication by its "rationalization factor" (RF). The RF may be complicated, as it may consist of as many as N—1 surds, where N is the product of the different indices of the given expression. Here the indices were 2, 3, and 5, so the RF could (and did) consist of 29 surds plus a rational quantity involving 30 coefficients ranging from 1770832 to 120355058. Determination of these values required the solution in integers of 29 equations in 30 unknowns. The detail of the solution is:

The problem is to rationalize the denominator of

$$\frac{1}{\sqrt{2}-\sqrt[3]{3}-\sqrt[5]{5}} = \frac{1}{r-s-t}$$

The procedure is to multiply both numerator and denominator by its "rationalization factor" consisting of 30 (= 2 × 3 × 5) terms, each consisting of a rational coefficient times a surd. The surds will be all possible combinations of the three given surds to powers less than the radical indices. To simplify typing, the given surds will be represented by r, s and t — as shown. The 30 coefficients will be represented by letters — unknowns to be evaluated. This rationalization factor will be multiplied by the given denominator, in a tabular array, deriving "product coefficients" for each of the surds. Each product coefficient will be equated to zero, so that the complete product will be the product coefficient of the rational unit. With just one rational unit, there will result 29 equations in 30 unknowns. These may be solved for 29 unknowns in terms of the 30th, or for all 30 in integers proportional thereto.

Rationalization factor terms		Product Coefficients	Eq No.
Coef.	Surd	·r ·s ·t	
A	1	2B − 3D − 5H = den	(1)
B	r	A − 3J − 5N = 0	(2)
C	s	2I − A − 5R = 0	(3)
D	s^2	2J − C − 5V = 0	(4)
E	t	2K − 3S − A = 0	(5)
F	t^2	2L − 3T − E = 0	(6)
G	t^3	2M − 3U − F = 0	(7)
H	t^4	2N − 3V − G = 0	(8)
I	rs	C − B − 5Z = 0	(9)
J	rs^2	D − I − 5d = 0	(10)
K	rt	E − 3a − B = 0	(11)
L	rt^2	F − 3b − K = 0	(12)
M	rt^3	G − 3c − L = 0	(13)
N	rt^4	H − 3d − M = 0	(14)
Ø	st	2W − E − C = 0	(15)
P	st^2	2X − F − Ø = 0	(16)
Q	st^3	2Y − G − P = 0	(17)
R	st^4	2Z − H − Q = 0	(18)
S	s^2t	2a − Ø − D = 0	(19)
T	s^2t^2	2b − P − S = 0	(20)
U	s^2t^3	2c − Q − T = 0	(21)
V	s^2t^4	2d − R − U = 0	(22)
W	rst	Ø − K − I = 0	(23)
X	rst^2	P − L − W = 0	(24)
Y	rst^3	Q − M − X = 0	(25)
Z	rst^4	R − N − Y = 0	(26)
a	rs^2t	S − W − J = 0	(27)
b	rs^2t^2	T − X − a = 0	(28)
c	rs^2t^3	U − Y − b = 0	(29)
d	rs^2t^4	V − Z − c = 0	(30)

Deductions:

Source		Eq. No.
3,2,26	2I = 3J + 10N + 5Y	(31)
4,9,30	2J = B + 10Z + 5c	(32)
5,27,2	2K = 3W + 6J + 5N	(33)
6,28,11	2L = 3X + 6a + B	(34)
7,29,12	2M = 3Y + 6b + K	(35)
8,30,13	2N = 3Z + 6c + L	(36)
15,11,9	2W = 3a + 2B + 5Z	(37)
16,12,23	2X = 3b + 2K + I	(38)
17,13,24	2Y = 3c + 2L + W	(39)
18,14,25	2Z = 3d + 2M + X	(40)
19,23,10	2a = K + 2I + 5d	(41)
20,24,27	2b = L + 2W + J	(42)
21,25,28	2c = M + 2X + a	(43)
22,26,29	2d = N + 2Y + b	(44)

So far, 15 unknowns have been eliminated, leaving 14 equations in 15 unknowns. Now substitute for B, L, W, a and b:

32	B = 2J − 10Z − 5c	(45)
36	L = 2N − 3Z − 6c	(46)
36,39	W = 2Y − 4N + 6Z + 9c	(47)
43	a = 2c − M − 2X	(48)
44	b = 2d − N − 2Y	(49)
33,47	2K + 7N = 6J + 6Y + 18Z + 27c	(50)
34,45,46	2J + 19c = 4N + 6M + 9X + 4Z	(51)
35,49	K + 3Y + 12d = 2M + 6N + 12Y	(52)
37,45,48	4J + 8N = 3M + 6X + 4Y + 27Z + 22c	(53)
38,49	2K + I + 6d = 3N + 2X + 6Y	(54)
41,48	2I + K + 2M + 4X + 5d = 4c	(55)
42,46,47	J + 8Y + 9Z + 12c = 4N + 4d	(56)

These seven with 31 and 40 make nine equations in 10 unknowns. Now substitute for I, K, Z:

$$2I = 3J + 10N + 5Y \quad (31)$$
$$2Z = 2M + X + 3d \quad (40)$$
$$K = 2M + 6N + 9Y - 12d \quad (52)$$

40,50,52	6J + 14M − 19N + 9X − 12Y + 27c + 51d = 0	(57)	6 equations in 7 unknowns; eliminate J by combination
40,51	2J − 10M − 4N − 11X + 19c − 6d = 0	(58)	
40,53	8J − 60M + 16N − 39X − 8Y − 44c − 81d = 0	(59)	
31,52,54	3J + 8M + 28N − 4X + 29Y − 36d = 0	(60)	
31,52,55	3J + 4M + 16N + 4X + 14Y − 4c − 7d = 0	(61)	
40,56	2J + 18M − 8N + 9X + 16Y + 24c + 19d = 0	(62)	
57 − 2 × 61	6M − 51N + X − 40Y + 35c + 65d = 0	(63)	5 equations in 6 unknowns; eliminate X by combination
4 × 58 − 59	20M − 32N − 5X + 8Y + 120c + 57d = 0	(64)	
60 − 61	4M + 12N − 8X + 15Y + 4c − 29d = 0	(65)	
62 − 58	28M − 4N + 20X + 16Y + 5c + 25d = 0	(66)	
58 + 2 × 62 − 2 × 61	18M − 52N − X + 4Y + 75c + 46d = 0	(67)	
4 × 64 + 66	108M − 132N + 48Y + 485c + 253d = 0	(68)	4 equations in 5 unknowns; eliminate M by combination
63 + 67	24M − 103N − 36Y + 110c + 111d = 0	(69)	
6 × 63 + 64	50M − 287N − 192Y + 295c + 382d = 0	(70)	
8 × 63 + 65	52M − 396N − 305Y + 284c + 491d = 0	(71)	
69 + 12(70 − 71)	1205N + 1320Y + 242c − 1197d = 0	(72)	3 equations in 4 unknowns; eliminate Y by combination
68 + 2 × 70 − 4 × 71	878N + 884Y − 61c − 947d = 0	(73)	
2 × 70 − 71 − 2 × 69	28N − 7Y + 86c + 51d = 0	(74)	
7 × 72 + 1320 × 74	45395N + 115214c + 58941d = 0	(75)	2 equations in 3 unknowns; eliminate N by combination
7 × 73 + 884 × 74	30898N + 75597c + 38455d = 0	(76)	
30898 × 75 − 45395 × 76	128156357c + 75494293d = 0	(77)	GCD = 7 coefficients relatively prime
77/7	18308051c + 10784899d = 0	(78)	

	d	= 18308051	Take c and d as least integers satisfying 78
	c	= −10784899	
75,76	N	= 3601201	Substituting back to derive other coefficients, testing each equation
72,73,74	Y	= 15291845	
68,69,70,71	M	= 3148973	"
63-67	X	= −36110637	"
57-62	J	= −18280543	"
52	K	= −54164855	From substitution equations, testing 50-56
40	Z	= 12555731	" "
31	I	= 28814803	" "
49	b	= 2431211	" testing 31-44
48	a	= 47502503	" "
47	W	= −5550819	" "
46	L	= 34244603	" "
45	B	= −108193901	" "
30	V	= 1770832	From basic equations, testing all others
29	U	= 17723056	"
28	T	= 11391866	"
27	S	= −23831362	"
26	R	= 18893046	"
25	Q	= −32961664	"
24	P	= 28693784	"
23	Ø	= −25350052	"
19	D	= 120355058	"
18	H	= 58073126	"
17	G	= 1889906	"
16	F	= −46871222	"
15	C	= −45415246	"
11	E	= 34313608	"
5	A	= −36835624	"
1	Denominator	= −867818606	From basic relation—not self-checking

Numerator = A + Br + etc. = $-36835624 - 108193901\sqrt{2} - 45415246\sqrt[3]{3} + 120355058\sqrt[3]{9}$ etc.

In summary, the result was

$$\frac{1}{\sqrt{2}-\sqrt[3]{3}-\sqrt[5]{5}} = \frac{-\text{RF}}{867\,818\,606}$$

in which RF is

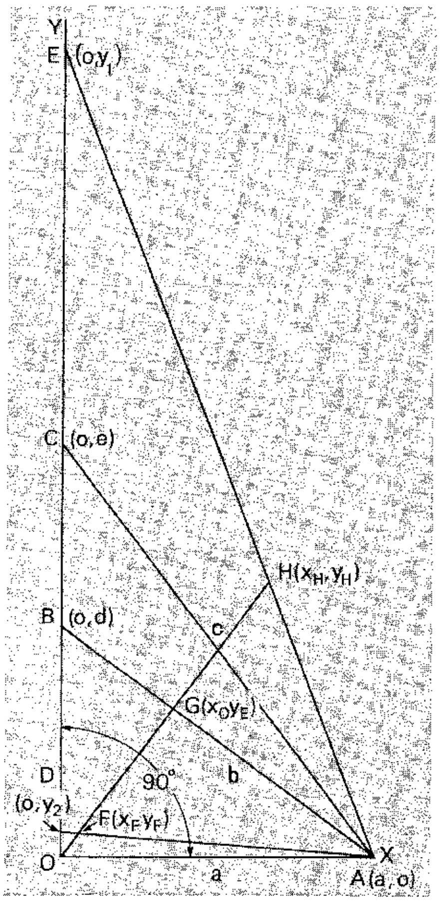
$$\begin{aligned}
&-36835624 - 108193901\sqrt{2} - 45415246\sqrt[3]{3} + 120355058\sqrt[3]{9} \\
&+ 34313608\sqrt[5]{5} - 46871222\sqrt[5]{25} + 1889906\sqrt[5]{125} + 58073126\sqrt[5]{625} \\
&+ 28814803\sqrt{2}\sqrt[3]{3} - 18280543\sqrt{2}\sqrt[3]{9} - 54164855\sqrt{2}\sqrt[5]{5} + 34244603\sqrt{2}\sqrt[5]{25} \\
&+ 3148973\sqrt{2}\sqrt[5]{125} + 3601201\sqrt{2}\sqrt[5]{625} - 25350052\sqrt[3]{3}\sqrt[5]{5} \\
&+ 28693784\sqrt[3]{3}\sqrt[5]{25} - 32961664\sqrt[3]{3}\sqrt[5]{125} + 18893046\sqrt[3]{3}\sqrt[5]{625} \\
&- 23831362\sqrt[3]{9}\sqrt[5]{5} + 11391866\sqrt[3]{9}\sqrt[5]{25} + 17723056\sqrt[3]{9}\sqrt[5]{125} \\
&+ 1770832\sqrt[3]{9}\sqrt[5]{625} - 5550819\sqrt{2}\sqrt[3]{3}\sqrt[5]{5} - 36110637\sqrt{2}\sqrt[3]{3}\sqrt[5]{25} \\
&+ 15291845\sqrt{2}\sqrt[3]{3}\sqrt[5]{125} + 12555731\sqrt{2}\sqrt[3]{3}\sqrt[5]{625} + 47502503\sqrt{2}\sqrt[3]{9}\sqrt[5]{5} \\
&+ 2431211\sqrt{2}\sqrt[3]{9}\sqrt[5]{25} - 10784899\sqrt{2}\sqrt[3]{9}\sqrt[5]{125} + 18308051\sqrt{2}\sqrt[3]{9}\sqrt[5]{625}
\end{aligned}$$

In connection with his solution, Mr. Rowe notes that an algebraic expression and its RF are complementary. Thus, if this RF had been given, its RF would have been computed the same way, with 30 unknowns, deriving that 27 were 0 and three were 1—as in the given denominator. "As a commentary on solving 29 simultaneous equations the hard way," Mr. Rowe writes, "I note that, being retired, I had lots of time. But were I not retired, I would have programmed the job for a one-minute task on an I.B.M."

Also solved by J. Q. Longyear and Walter Penney.

20 Construct a triangle given: 1. The bisector of side c; 2. The altitude on side c; and 3. The bisector of the angle opposite side c.

The following solution was submitted by William Burgess:



Given OA $= a$, OB $= b$, and OC $= c$, the problem is to construct triangle APE so that EC $=$ CD, angle DAB $=$ angle BAE, and OA is perpendicular to DE: $b > a$ and $c > b$. Lay out a along the x axis and find points B and C.

Let OB $= d(\sqrt{b^2 - a^2})$.

Let OC $= e(\sqrt{c^2 - a^2})$.

From C strike an arc of radius

$\sqrt{e^2 - a^2 + (e/d)\sqrt{a^2 - d^2}}$,

striking OY at D and E. Triangle ADE is the required triangle.

To find the length of

$\sqrt{e^2 - a^2 + (e/d)(a^2 - d^2)}$

consider that this term may be expressed as

$$\left(\sqrt{e^2 - a^2} + \sqrt{(e/d)(a^2 - d^2)}\right)^2 - \left(\sqrt{2\sqrt{e^2 - a^2}\,\sqrt{(e/d)(a^2 - d^2)}}\right)^2$$

Let

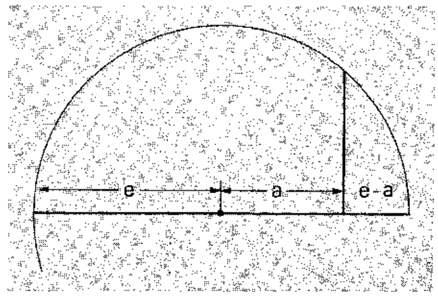
$\sqrt{e^2 - a^2} = 1$,

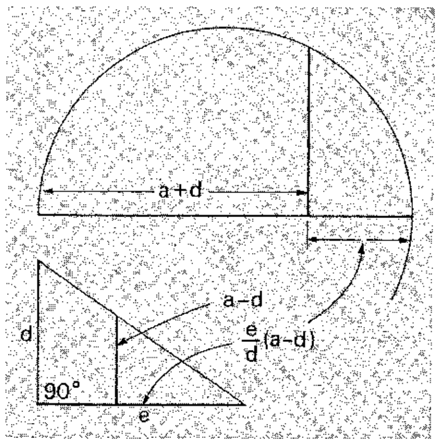
$\sqrt{(e/d)(a^2 - d^2)} = 2$,

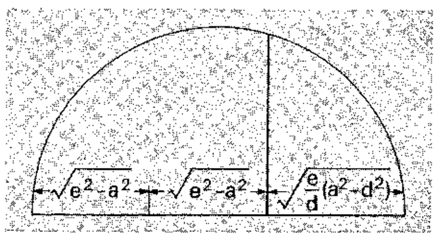
$\sqrt{2\sqrt{e^2 - a^2}\,\sqrt{(e/d)(a^2 - d^2)}} = 3$;

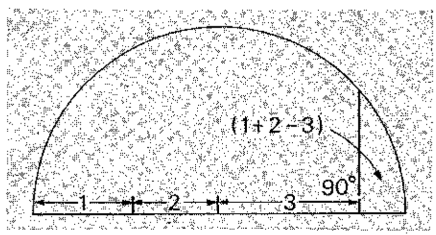
then $e^2 - a^2 + (e/d)(a^2 - d^2)$

$= (1 + 2 + 3)(1 + 2 - 3)$.









To prove that ADE is the required triangle, consider:

1. a is the altitude on side DE, by construction.
2. c bisects side DE, by construction.
3. To prove that angle DAB $=$ angle DAE, draw line OH perpendicular to line AB, intersecting AD at $F(x_F, y_F)$, AB at $G(x_G, y_G)$ and AE at $H(x_H, y_H)$.

Assume that FG $=$ GH; then BA bisects angle DAE as required. We know C is the midpoint of DE; therefore $y_1 + y_2 = 2e$. We also assume $y_F + y_H = 2y_G$. We can then write equations of the lines as follows:

AD: $x/a + y/y_2 = 1$

AB: $x/a + y/d = 1$

AE: $x/a + y/y_1 = 1$

OH: $x/d = y/a$.

Solve for coordinates of points F, G, and H, finding:

$y_F = (a^2y_2)/(a^2 + y_2d)$

$y_G = a^2d/b^2$

$y_H = (a^2y_1)/(a^2 + y_1d)$

Set $2y_G = y_F + y_H$ and solve for y_1 and y_2: gather the $(y_1 + y_2)$ terms and y_1y_2 terms, remembering that $y_1 + y_2 = 2e$ and that $y_1 = 2c - y_2$. This will give a quadratic equation in y_2 which is in the form

$Ay_2^2 + By_2 + C = 0$.

Using $y_2 = (-B \pm \sqrt{B^2 - 4AC})/2A$ and simplifying,

y_2 will be found as

$e \pm \sqrt{e^2 - a^2 + (e/d)(a^2 - d^2)}$;

y_2 is the negative value and y_1 is the positive. This is per construction.

Also solved by Steve Deutsch, Raymond Gaillard, R. Robinson Rowe, and John T. Rule.

Better Late Than Never

Solutions to problems published earlier have come from readers as follows:

2 Michael Rolle and J. Q. Longyear

5 Several solutions were received immediately after publication in addition to Winslow H. Hartford's, published in February. The other names were inadvertently omitted: Jerry Blum, Harold Donnelly, Thomas B. Jabine, Hubert duB. Lewis, Mrs. Martin S. Lindenberg, E. C. Signorelli, and the proposer, Smith D. Turner.

6 and **7** Charles Bures and Michael Rolle

8 The Olive Drab Phantom

11 Hugh D. Sims

12 Robert W. Baird and Paul Schweitzer

13 Mark Baldwin and John T. Rule

14 Charles Bures

15 Charles Bures, John T. Rule, Paul Schweitzer, and Steven Winkler.

Allan J. Gottlieb studied mathematics at M.I.T. with the Class of 1967, and he is now pursuing advanced study as a teaching assistant at Brandeis University. Send solutions, problems, and comments to him at the Department of Mathematics, Brandeis University, Waltham, Mass. 02154.

April Tech-Crostic Solution

Drugs for injection are necessarily in solution and solutions which are not sterile, isotonic, and neutral in pH, or nearly so, are damaging. Drugs taken by mouth may be solutions, suspensions, or solids, usually made into tablets or enclosed in capsules.

—C. R. B. Joyce, *Psychopharmacology.*