Allan J. Gottlieb

An Ecological Setting

I am considering a small change in format. Currently we reserve one problem per month for bridge fans. But perhaps chess and computer enthusiasts should not be slighted. If there is interest, a rotating plan involving one or two of the problems per month could be arranged. Due to considerations of space and time, we are limited to five problems per issue to which solutions are printed, so any increase in reserved slots for chess, bridge, computing, etc., means a corresponding reduction in the number allotted to everything else. (The number of "speed"—i.e., unanswered—problems given is quite flexible.)

Readers who have suggestions or opinions on this matter, please write. Your inclinations will be very influential.

Problems

Our bridge problem is from Paul D. Berger:

26 Given the following hands, South the successful bidder at 4 spades, and the opponents' lead of ♣ 7:

North:
♠ K
♥ 9 5 4 2
♦ K 7
♣ K Q J 8 6 5

West:
♠ 8 6
♥ K J 10 6 3
♦ A J 9 6 3
♣ 7

East:
♠ 7 4
♥ A Q 8
♦ Q 4 3
♣ A 10 4 3 2

South:
♠ A Q J 10 9 5 3 2
♥ 7
♦ 10 8 2
♣ 9

Do you choose offense or defense?

An old favorite in an ecological setting from John E. Prussing:

27 Three hoboes spent the day gathering aluminum cans to sell back to a can company for ½¢ each. That night they left the cans in a pile to be divided equally in the morning. During the night one hobo decided he wanted his share then. He divided the cans into three equal piles and had one odd can left over. He took his pile, pushed the other two piles back together, threw the odd can on the big pile, and stole away into the night with his cans. Later another of the hoboes did the same thing: three equal piles, one old can left over, took his share, threw the odd can on the big pile, ran away. Still later the third hobo did the same thing and had the same experience with one can. After the hoboes had left, the number of cans in the big pile was exactly divisible by three. In the morning a fourth hobo found the pile left behind by the others and sold them to the company. How much money did he receive? (Hint: it was less than 25¢.)

John (Boog) Rudy sends to us the following real-life problem:

28 While working on learning curve formulations I came up with
$A + 1 = N^{1-B} + AB$
I must solve this for B. I know that there is only one solution in the range $0 < B < 1$, for it is the answer to a real-life problem. How do I solve it exactly?

Stephen Kent submits the following problem which he encountered in a mathematics contest:

29 In how many different ways can eight numbers be rearranged such that no number occupies its original position? Check your answer by writing out all the possibilities. Next, find the answer for n numbers in general.

We conclude this month's offering with Charles Heiberg's analysis problem:

30 Given the notations
$f: R \rightarrow R$,
$\Delta_t f(x) = f(x + t) - f(x)$,
show that there exists f continuous such that
$\lim_{x \rightarrow +\infty} \Delta_t f(x) = 0$ for all rational t; and
$\lim_{x \rightarrow +\infty} \Delta_t f(x)$ does not exist for almost all (measure-theoretic) t.

Speed Department

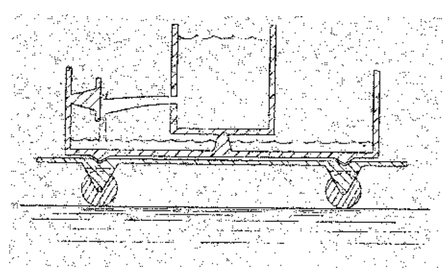
Frank Rubin sends us the following:

SD9 A flag pole erected in the center of a rectangular field is discovered to lean at an unknown angle. Bracing ropes from the corners of the field to the midpoint of the flagpole are found to have lengths of a, b, c, and x, going clockwise around the field. Find x as a function of a, b, and c.

Donald E. Savage says he "seems to get the message" that we need some "speed" problems, giving citations to four 1970 issues of *Technology Review*. Here is his, which he says "really is a 'speed' problem (intentional pun)":

SD10 It is often stated that for Newtonian physics one nonaccelerating frame of references is as good as another. Consider the case of a passenger in a train moving (and therefore not the Penn Central) with velocity V_t. Beside the track is a post on which a gun is mounted. The gun is aimed parallel to the track and in the opposite direction from that in which the train is moving. The gun's muzzle velocity is v. Thus, from the passenger's point of view, firing the gun raises the bullet's velocity from V_t to $V_t + v$. Thus the kinetic energy of the bullet is increased by
$\frac{1}{2}m(V_t + v)^2 - \frac{1}{2}m(V_t)^2$
$= \frac{1}{2}mv^2 + mV_tv$,
where m is the mass of the bullet. Since this energy came from the gunpowder, it appears that the energy available from the powder depends upon how fast the observer is moving.

Let's end by violating Newton's Laws—a problem in the September, 1970, issue of The *Physics Teacher* and there credited to Lewis Epstein:



SD11 We are all familiar with mechanisms supposed to violate the first law of thermodynamics, and some of us have even made acquaintances with mechanisms supposed to violate the second law of thermodynamics, but have you ever met one which presumes to violate Newton's first or second law? The mechanism about to be described is in-

tended to accelerate without the application of external force or the ejection and loss of reaction mass. The mechanism does require energy to operate.

A sled- or railroad-mounted prototype of the mechanism will be outlined; however, its immediate adaptation to space craft is apparent. The mechanism is truly the essence of simplicity. (Perhaps this is why it has not been previously noticed.) It consists of a water bucket with a hole punched in its side close to the bottom. The jet of water squirting from the hole impinges on a splash plate and then falls into a collecting basin. If desired, the cycle can be closed by a pump that lifts the water from the basin back into the bucket.

Since the prototype is railroad-mounted, only horizontal forces are of concern. Were the bucket not punctured, there would be no net horizontal force on it, since the horizontal pressure exerted by the water at any point on its side would be nullified by an equal but counter-directed pressure on the opposite side of the bucket. Punching a hole in the bucket upsets this balance. If a hole of effective area A is opened at a depth where the water pressure is P, the balance is upset by an amount PA, representing a net force F_b pushing on the inside of the bucket opposite the hole. So we have an elementary rocket.

Now we must consider the force, F_s, of the jet impinging on the splash plate. Offhand F_s would be expected to counteract the reaction force, $F_b = PA$, on the bucket. Let us see. In accordance with Torricelli, if energy is to be conserved, the velocity, v, of the water squirting from the hole must equal the velocity that a body would obtain when falling freely from the height, h, of the water surface to the hole. Whence:

$\frac{1}{2} v^2 = gh.$

Now, if the water has density, ρ, its pressure at the depth where the hole was punched must be:

$P = \rho gh$

and thus:

$P = \rho \frac{1}{2} v^2$

or:

$F_b = \rho \frac{1}{2} v^2 A.$

The counter force against the splash plate must be the time rate of change of the water jet's momentum as it strikes the plate. To make things simple, the plate is covered with a screen, so the water will not rebound. The plate then simply absorbs the momentum in the jet. The mass of water impinging on the plate during a time, t, must be ρvAt and its momentum must be $\rho v^2 At$. The time rate of change of this momentum is $\rho v^2 A$ and accordingly the force against the splash plate is:

$F_s = \rho v^2 A.$

So it turns out that:

$F_s = 2F_b.$

The force on the splash plate does not just counteract the reaction force on the bucket, it overwhelms it by a factor of two, and we must conclude there is a net force on the whole mechanism. The mechanism is compelled to accelerate in the direction of the splash plate.

What consequences this may foreshadow for interplanetary travel cannot be known.

Solutions

11 Given the hands shown and the bidding as listed,

North: ♠ 8 6; ♥ A K x; ♦ Q 9 7; ♣ A 10 9 x x

West: ♠ A Q 10 x x; ♥ Q J 10 x x; ♦ x; ♣ J x

East: ♠ J 9; ♥ 9 x x x; ♦ x x x; ♣ K Q x

South: ♠ K x x; ♥ x; ♦ A K J 10 8 5; ♣ x x x

South	*West*	*North*	*East*
1 diamond	1 spade	2 clubs	pass
2 diamonds	2 hearts	3 hearts	pass
3 no-trump	4 hearts	5 diamonds	pass
pass	pass		

and West's lead of the ♥ Q, show that if the declarer wins the first trick with the ♥ A then he must lose two spade tricks and a club when East gets in with the ♣ K.

The following is from John W. Meader: The only sure way to make 11 tricks is to ruff out the club suit. In order to do this without giving up a club to East, which would be fatal because of the spade return, South must find two club discards. He ducks the opening heart; wins the next lead—say a club (the best defense)—; draws one trump with the ♦ Q, throwing the ♦ J or ♦ 10 under it; discards two clubs on the top hearts; ruffs a club high; returns to dummy with a trump; ruffs another club high; wins a third diamond North; discards two spades on the last two clubs; and gives up a spade.

Also solved by Philip D. Bell, Winslow H. Hartford, E. C. Ingraham, R. Robinson Rowe, and the proposer, John Rudy.

12 Show that for every odd positive integer n, sin nx can be expressed in the form $\sin nx = a_1 \sin x + a_3 \sin^3 x + \ldots + a_n \sin^n x$ and derive a general formula for the coefficients a_k.

Gilbert Shen supplies the following calculation:

$$\sin nx = \text{Im}\,(\cos x + i \sin x)^n$$

$$= \text{Im} \sum_{k=0}^{n} (\cos^{n-k} x)(i \sin x)^k \binom{n}{k}.$$

The imaginary part of the R.H.S. consists of the sum of the odd k terms:

$$\sin nx = \sum_{k\ \text{odd}}^{n} \binom{n}{k} (\cos^{n-k} x)$$

$(\sin^k x) i^{k-1}.$

Since both k and n are odd, n − k is even. Hence

$$\sin nx = \sum_{k\ \text{odd}}^{n} \binom{n}{k} (-1)^{(k-1)/2}$$

$\sin^k x (1 - \sin^2 x)^{(n-k)/2}$

which is manifestly a polynomial in sin x. (Note that $(k-1)/2$ and $(n-k)/2$ are integers.) The highest power in this polynomial is the term $(\sin^k x)(\sin^2 x)^{(n-k)/2}$ when k = n. This is just $\sin^n x$. Since sin nx is odd in x, we retain only the odd-powered terms in the expansion. Hence

$$\sin nx = \sum_{k\ \text{odd}}^{n} a_k \sin^k x.$$

To obtain an expression for a_k, define $y \equiv \sin x$. Then

$\sin nx = \sin(n \sin^{-1} y) \equiv f(y)$

is a factor of y. Taylor expand about y = 0:

$$f(y) = \sum_{k=0}^{\infty} \left.\frac{f^{(k)}(y)}{k!}\right|_{y=0} y^k$$

By comparison, it must be that

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!} \left(\frac{d}{d \sin x}\right)^k \sin nx$$

at x = 0.

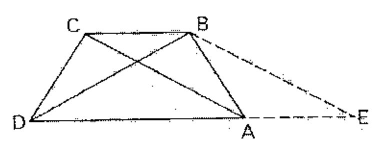
Also solved by Harold Donnelly, Winslow H. Hartford, Ivar and Carolyn Kist, R. Robinson Rowe, Donald E. Savage, Mark Schoenberg, and the proposer, Arthur W. Anderson.

13 Given a convex quadrilateral ABCD with diagonals AC and BD, and given that AC = BD

Angle BAC = angle CAD

Angle CBD = angle BDA,

prove that the quadrilateral is a trapezoid.



Robert Pogoff submitted the following geometrical proof, making the assumption that the problem is to prove that the quadrilateral is an *isosceles* trapezoid (or a square, which—as he says—obviously meets the requirements of the given conditions):

The problem is to prove CB ∥ DA and BA = CD.

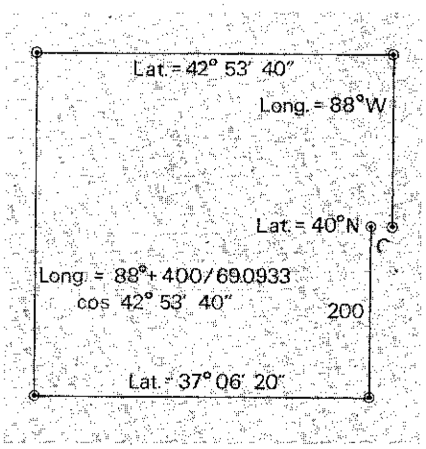
1. Draw BE ∥ CA
2. Extend DA to E
3. DE ∥ CB (angle CBD = angle BDA)
4. Angle BEA = angle CAD (BE ∥ CA)
5. Therefore CBEA is a parallelogram (two pairs of parallel sides)
6. Therefore BE = CA (opposite sides of a parallelogram)
7. CA = BD (given)
8. Therefore BE = BD
9. Therefore angle BEA = angle BDA (base angles of an isosceles triangle)
10. Therefore angle CAD = angle BDA
11. DA = DA
12. Therefore △CAD ≅ △BDA
13. Therefore BA = CD

If angle BAD is a right angle, then it follows that all the acute angles are 45°, all the triangles are isosceles, and CB = BA = AD = DC; then ABCD is a square. Note that the given fact, that angle BAC = angle CAD, is not necessary to prove that the quadrilateral is an isosceles parallelogram; nor is the term *convex;* it must be convex if angle CBD = angle BDA.

Also solved by Winslow H. Hartford, E. C. Ingraham, John L. Joseph, Mrs. Martin S. Lindenberg, Roger Milkman, R. Robinson Rowe, John Rudy, and Gilbert Shen.

14 Starting from a point 40° N. 88° W., a man walked 200 miles due north, then 400 miles due west, 400 miles south, 400 miles east, and finally 200 miles north. To his amazement, he was not at his starting point. How far away was he?

R. Robinson Rowe took care of this one rather easily:



Due to convergence of meridians, the man did not walk around a square but followed the path in the drawing. A precise solution on the Clarke spheroid would be very complicated, but it should be near enough to use a sphere with the geoid's average radius of 3958.794 miles. He will end at 40° N. latitude but west of his startnig point. Letting the latitudes of his second and fourth courses be N and S, the terminal gap G was
$G = 400 \cos 40° (\sec N - \sec S)$
$= 34.0449$ miles.

Also solved by E. W. Beohne, Harold Donnelly, Harry V. Ellis, III, David H. Geisler, Winslow H. Hartford, Mrs. Martin S. Lindenberg, Gilbert Shen, Lawrence N. Smith, and J. Supine.

15 Given the following, find x and y in terms of a and c.
Jules Sandock submitted the following exhaustive solution:
Let $A = a/c$, $Y = y/c$, and $X = x/c$.
It is clear that $v = y - a$, $u = x - a$.
$v/a = a/u$
Let $S = u/a$, $T = v/a$. Then
$ST = 1$.
$1 = X^2 + Y^2 = (u + A)^2 + (v + A)^2$
$= u^2 + v^2 + 2A(u + v) + 2A^2$
$1/A^2 = S^2 + T^2 + 2(S + T) + 2$
$S^2 + T^2 = (S + T)^2 - 2ST$
$= (S + T)^2 - 2$
$1/A^2 = (S + T)^2 + 2(S + T)$.
Let $R = S + T$; then
$1/A^2 = R^2 + 2R$
$1/A^2 + 1 = (R + 1)^2$
$R + 1 = \sqrt{1 + 1/A^2}$
$R = \sqrt{1 + 1/A^2} - 1$
It is given that $S + T = R$
and that $ST = 1$.
$S = R - T = R - 1/S$
$S^2 = RS - 1$
$S^2 - RS = -1 = (S - r/2)^2 - R^2/4$
$(S - R/2)^2 = R^2/4 - 1 = (R^2 - 4)/4$.
From the figure, $x > y$, $u > v$, and $S > T$; so $S > R/2$ and $T < R/2$.
$R = S + 1/S > z$ (unless $S = 1$, $T = 1$, and $v = u$)
$R^2 - 4 > 0$
$S - R/2 = \sqrt{R^2 - 4/2}$
$S = ½\{R + \sqrt{R^2 - 4}\}$
$T - R/2 = -\sqrt{R^2 - 4/2}$
$T = ½\{R - \sqrt{R^2 - 4}\}$
$X = u + A = A(S + 1)$
$= A/2\{R + 2 + \sqrt{R^2 - 4}\}$
$= A/2\sqrt{R + 2}\,\{\sqrt{R + 2} - \sqrt{R - 2}\}$
$Y = v + A = A(T + 1)$
$= A/2\{R + 2 - \sqrt{R^2 - 4}\}$
$= A/2\sqrt{R + 2}\,\{\sqrt{R + 2} - \sqrt{R - 2}\}$
$R + 2 = \sqrt{1 + 1/A^2} + 1$
$R - 2 = \sqrt{1 + 1/A^2} - 3$

$$X = A/2\sqrt{\sqrt{1 + 1/A^2} + 1}\left\{\sqrt{\sqrt{1 + 1/A^2} + 1} + \sqrt{\sqrt{1 + 1/A^2} - 3}\right\}$$

$$Y = A/2\sqrt{\sqrt{1 + 1/A^2} + 1}\left\{\sqrt{\sqrt{1 + 1/A^2} + 1} - \sqrt{\sqrt{1 + 1/A^2} - 3}\right\}$$

$x = cX$ and $a = cA$, so

$$x = a/2\sqrt{\sqrt{1 + c^2/a^2} + 1}\left\{\sqrt{\sqrt{1 + c^2/a^2} + 1} + \sqrt{\sqrt{1 + c^2/a^2} - 3}\right\}$$

$$y = a/2\sqrt{\sqrt{1 + c^2/a^2} + 1}\left\{\sqrt{\sqrt{1 + c^2/a^2} + 1} - \sqrt{\sqrt{1+c^2/a^2}-3}\right\}$$

$c/a = 2\sqrt{2}$

$$x = a/2\sqrt{\sqrt{1 + 8} + 1}\left\{\sqrt{\sqrt{1 + 8} + 1} + \sqrt{\sqrt{1 + 8} - 3}\right\}$$

$$= a/2\sqrt{4}\,\{\sqrt{4} + u\} = a/2\,(2)(2) = 2a.$$

Also solved by William Burgess, Harold Donnelly, Arup Dravid, Edward S. Gershuny, William Glassman, Winslow H. Hartford, John W. Meader, R. Robinson Rowe, John Rudy, Jules Sandock, Donald E. Savage, Gilbert Shen, J. J. Sytek, and the proposer, John L. Sampson.

Better Late Than Never

Two communications concerning problems in Volume 72 have been received; A. Porter has submitted a solution for problem **40,** and George O. Smith makes the following significant contribution on problem **43,** to which "so-called" (his word) solutions were accepted in the January issue:
I cry foul! You have violated one of the most sacred tenets in geometry by permitting the appearance of irrationals in the argument. Irrationals are proscribed —and aside from a few notable coincidences, the transcendental functions are irrational. Thus your "solutions" are no more than approximations. Besides, the argument is more elegant if we observe:
1. The five points of the star lie on a circle and are equidistant.
2. Therefore, all acute angles are equal; all obtuse angles are equal; and all triangles are isosceles.
3. Angle AEB = angle BEF = angle EFA (either base is twice the vertex angle).
4. Thus the acute triangle DAF is that special case in which
DF/AD = AD/(DF + AD),
which is the so-called "golden mean," and thus the "width" AD is incommensurate with the "separation" DF.
5. Thus the problem cannot truly be solved, since the "golden mean" is itself irrational. The "golden mean," by the way, is 1.61803, the only "number" that becomes its own reciprocal simply by subtracting 1.000. The five-pointed star, also called either the "pentangle" or the "pentagram," is also the emblem of the Pythagorean Brotherhood, a mystico-religious organization that believed that all numbers were whole and which came unglued when one of the members proved, by argument, that the diagonal of a square is itself incommensurable with the side since it, too, is irrational. Such lines can be drawn, but they cannot be defined nor described in finite terms.

E. W. Boehne and William Burgess also responded to problem **43.**

Solutions have come to problems in Volume 73 as follows:
6 Herbert Messenger, Albert Morris, and Patrick J. Sullivan.
7 Lance Wilson.
8 William Burgess.

Allan J. Gottlieb, who graduated from M.I.T. in 1967, teaches mathematics at Brandeis University. Send problems and solutions to him at the Department of Mathematics, Brandeis University, Waltham, Mass. 02154.