Riding the Railroad

My girl and I went to visit my old dormitory at M.I.T. tonight. It still looks nice—but not as good as I remember. When I walk down the corridors I can see all the familiar faces and hear all the familiar voices. All of a sudden I feel very old and nostalgic.

Enough of this foolishness.

Two requests: please refer to problems by number, not by name; and please send in bridge problems.

Problems

As is usual (we hope), we start with a bridge problem—this one by John Price.

16 Given the following hand:

North: ♠ Q x x x / ♥ Q x x / ♦ A K x / ♣ K 9 x

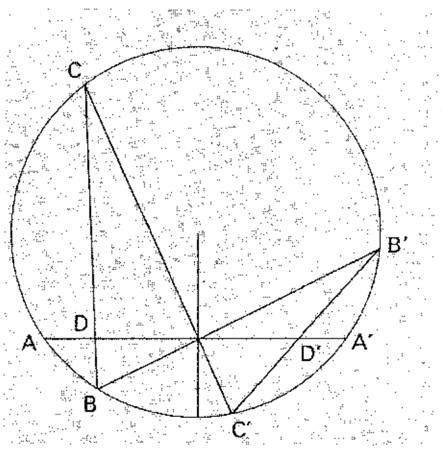
West: ♠ 10 9 x / ♥ A / ♦ x x x / ♣ A Q 10 x x x

East: ♠ J / ♥ K J 10 x x x / ♦ J x x x / ♣ J x

South: ♠ A K x x x / ♥ x x x / ♦ Q x x / ♣ x x

West opens with the ♥A and shifts to a low club. Can you make the contract—four spades?

Our second problem is from Roy G. Sinclair:



17 Let the chord AA' perpendicular to a radius of a circle intersect the radius in M. Let any other two chords through M intersect the circle in B,B' and C,C'. Let BC and B'C' intersect AA' in D and D'. Show that MD = MD'.

Russell A. Nahigian has a railroad problem (so do Long Island commuters):

18 A railroad operates under the following conditions: 1. There is exactly one train per day to take passengers from any given suburban station to any other suburban station; 2. No two trains in the same direction have more than one stop in common. 3. Each train stops at exactly three suburban stations; and 4. More than one train per day in each direction stops at each suburban station. How many trains per day are there in each direction and how many stations are there on the line?

19 Frank Rubin wants you to rationalize the denominator of

$$\frac{1}{\sqrt{2}-\sqrt[3]{3}-\sqrt[5]{5}}$$

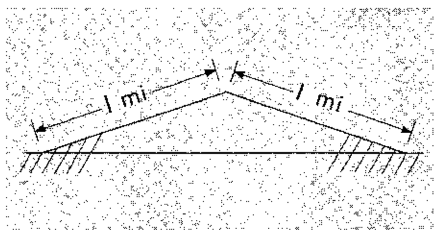
or prove it impossible.

We end with a geometry problem from Rüdiges Dierstein:

20 Construct a triangle given: 1. The bisector of side c; 2. The altitude on side c; and 3. The bisector of the angle opposite side c.

Speed Department

An easy one from James R. Bledsoe:



SD6 A car approaches a hill which is one mile long going up and one mile long down the other side. The car travels at 30 m.p.h. until reaching the summit. How fast must it descend the other side in order to average 60 m.p.h. if the total trip takes two minutes?

Solutions

1 Given the following hands, with the contract four spades:

North: ♠ x x x / ♥ x x x x x / ♦ A x / ♣ A K Q

West: ♠ x x x x / ♥ A K Q J x / ♦ x x x / ♣ x

East: ♠ x / ♥ x / ♦ K J 10 x x x / ♣ x x x x x

South: ♠ A K Q J 10 / ♥ x x / ♦ Q x / ♣ J x x x

West leads ♥K and ♥J and continues with ♥Q (his best play). Can you make the contract?

The following is from Lawrence Herman Shiller:

It is obvious that the declarer needs one more entry to his hand to cash his game-going trick, ♣J. This can be done in only one way: the declarer must ruff the ♥Q with the ♠10 and draw four rounds of trumps. On the fourth round, he must discard the ♦A from dummy. Next he cashes the ♣A, ♣K, and ♣Q and leads a low diamond to his ♦Q. If East goes up to ♦K, he will have only a diamond or a club to return and declarer's hand will be good; if East ducks, the declarer wins the ♦Q and cashes the ♣J, conceding the last trick to East. Either way the declarer makes 10 tricks, losing only two hearts and one diamond. (It is interesting to note that assuming one of North's small hearts is the ♥10, the N-S contract of three no-trump is unbeatable.)

2 Let N be some fixed positive integer. Show that there exist positive rational numbers $a_1, \ldots, a_N$ such that for any m, $1 \leq m \leq N$

$$S(m) = \sum_{i=1}^{m} a^3_i$$

is the square of a rational number, and $S(N) = 1$.

Apparently this problem was rather difficult. The only response was by R. Robinson Rowe, who seems to arrive at a dead end. In the hope that it may help someone else, I am reprinting his partial solution: It was my hunch that the summations involved the fact that the sum of the first m cubes was the square of the mth triangular number, that is,

$$\sum_{1}^{m} i^3 = [\tfrac{1}{2} m(m-1)]^2 = T_m^2.$$

But if $S(N) = 1$, each term of the series (all being positive) must be less than 1, suggesting that $a_i = ki$, where k is a rational fraction. This leads to
$S_m = k^3 T_m^2$,
which must be a square, so that k is a square, and to
$S_N = k^3 T_N = 1$, so that
$k^3 = u^6 = 1/T_N^2$.
However, except for the trivial $T_1 = 1$, no triangular number can be a cube. So my next hunch was that $a_i = ki = u^2 i$ for $N - 1$ terms, and a_N is not kN but chosen so that $k^3 T_{(n-1)}^2 - a_N^3 = 1$, which depends upon solution in integers of $X^6 = Y^3 + Z^6 T^2$. This may be soluble, but for the moment it is a dead end, too.

3 Pascal's triangle can also be written in rectangular form, in which case it looks like this:

1	1	1	1	1	...
1	2	3	4	5	...
1	3	6	10	15	...
1	4	10	20	35	...
.	.	.	.	.	
.	.	.	.	.	

The first row and first column consist entirely of 1's and the other entries are found by adding the number to the left and the number above. If this array is continued to n rows and n columns, where n is any positvie integer, prove that the determinant of the resulting matrix is 1.

The following is from Thomas H. Sadler: Let M be the nXn matrix in question. From M,I obtain a sequence of nXn matrices, $M = M_1, M_2, \ldots M_n$, such that $\det M_1 = \det M_2 = \ldots = \det M_n$, and M_n is in triangular form with all diagonal elements equal to 1. I get M_k from $M_{(k-1)}$ in the following way:
1. Rows $1, \ldots, (k-1)$ of M_k and $M_{(k-1)}$ are the same.
2. Row i, where $i = k, \ldots, n$ of M_k is simply row i of $M_{(k-1)}$ *minus* row $(i - 1)$ of $M_{(k-1)}$.
Since each row of M_k is the sum of the corresponding row of $M_{(k-1)}$ and a scalar multiple of another row of $M_{(k-1)}$, $\det M_k = \det M_{(k-1)}$. Thus $\det M_1 = \ldots = \det M_n$. The rules for Pascal's triangle give us
a. $M(i,1) = M(1,j) = 1$, $i, j = 1, \ldots, n$.
b. $M(i,j) = M(i,j-1) - M(i-1,j)$, $i, j = 2, \ldots, n$.
Using the procedure given earlier, I get M_2 from M:
1. $M_2(1,j) = 1$, $j = 1, \ldots, n$
2. $M_2(i,j) = M(i,j) - M(i-1,j) = M(i-1,j) + M(i,j-1) - M(i-1,j) = M(i,j-1)$ for $j = 2, \ldots, n$, $i \geqq 2$; and $M_2(i,1) = 0$ for $i = 2, \ldots, n$. So the first elements of rows $2, \ldots, n$ of M_2 are zeros and elements $2, \ldots, n$ of each of these rows are just elements $1, \ldots, n-1$ of the corresponding rows of M. Thus rows $2, \ldots, n$ of M_2 have the relationship b. above. Now, hopefully, you see that doing this over and over results in row i of M_n having elements $i, \ldots, n$ equal to elements $1, \ldots, (n-1)$, respectively, of M and every other element zero. Thus M_n is in triangular form. But the determinant of a matrix in triangular form is the product of its diagonal elements. Since all diagonal elements of M_n are 1's, $\det M_n = 1$. Therefore, $\det M = \det M_n = 1$.

4 A magic square is a square matrix of numbers such that rows, columns, and diagonals all sum to the same total. Create a 5 x 5 magic square using 25 two-digit numbers composed of the digits 0, 1, 6, 8, and 9. The magic square must also work when turned upside down, so that 90 becomes 06, etc.

The following is from Loren L. Dickerson, Jr., who offers two magic squares which meet the conditions specified. He writes: The magic squares below are typical of 32 that can be made by the normal knight's move, starting from various cells of the grid, and having the central number of the series, 66, centered. Any distribution of the specified digits, 0, 1, 6, 8, 9, whereby each digit appears only once in the units and once in the tens in each row, column, and diagonal will make a magic square; and any pairs of digits may be interchanged, as the 6 and 9 are when inverted. Modifications of the knight's move, if they work, should produce only reproductions in a square as small as 5 x 5. These squares are pandiagonal in that all 10 diagonals of each have a sum of 264, both right-side-up and inverted. They cannot, however, be completely associated (all pairs of cells equidistant from the center summing to 99), because there is no complement to 6 in the series of digits, and they would not be associated after the inversion, anyway.

91	68	00	86	19
06	89	11	98	60
18	90	66	09	81
69	01	88	10	96
80	16	99	61	08

18	61	89	96	00
86	90	08	11	69
01	19	66	80	98
60	88	91	09	16
99	06	10	68	81

5 Determine a rational number whose square, when increased or decreased by 5, is still a square.

The following is from Winslow H. Hartford, who writes that the problem is "a real favorite of mine, perhaps because it was proposed to me by my boss on my first permanent job in 1934. I kept the job and solved it." Mr. Hartford's answer is 41/12, the square of which is 1681/144; the other numbers are 961/144 and 2401/144, which are $(31/12)^2$ and $(49/12)^2$, respectively.

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