Five Problems, Three Corrections

Hi. Since this column starts a new volume of *Technology Review,* let me briefly mention the ground rules for the benefit of new readers.

In each issue we'll publish five problems and two "speed" problems, selected from those submitted by readers. Three months later we'll publish an answer—as received from a reader—to each problem; but normally the "speed" problems remain unanswered.

Thus the column depends on reader response. None of the problems or solutions is mine, so all the credit (and blame) belongs to all of you out there in magazineland. As of now (August 13) I have a backlog of proposed problems which will last almost until the snow melts, so please be patient if your masterpieces don't appear for a while. "Speed" problems, on the other hand, are in short supply.

Before getting to the problems, let me clear up three points from previous columns. First, the magic punctuation is John when James had had "had had" had had "had" "had had" had had the teacher's approval. Apparently John shouldn't have changed James' "had had" to a mere "had." Second, while very pressed for time last spring I offered a lifetime subscription for anyone who could separate my sunsets by 42 hours. Since then many people have suggested that I go into earth orbit. Although this would surely solve the solid waste problem at my mother's house, I am not giving credit; the obvious intent of the problem was to increase my working hours. Finally, in the July/August issue I gave Gauss credit for proving the impossibility of constructing certain polygons with straight edge and compass. Michael Goldberg has pointed out my error and refers interested readers to the *American Mathematical Monthly,* Volume 75 (1968), p. 647.

Problems

We start the volume off right with a bridge problem from Winslow H. Hartford:

1 Given the following hands, with the contract four spades:

North:
♠ x x x
♥ x x x x x
♦ A x
♣ A K Q

West:
♠ x x x x
♥ A K Q J x
♦ x x x
♣ x

East:
♠ x
♥ x
♦ K J 10 x x x
♣ x x x x x

South:
♠ A K Q J 10
♥ x x
♦ Q x
♣ J x x x

West leads ♥K and ♥J and continues with ♥Q (his best play). Can you make the contract?

Frank Rubin submits the following:

2 Let N be some fixed positive integer. Show that there exist positive rational numbers $a_1, \ldots, a_N$ such that for any m, $1 \leq m \leq N$

$$S(m) = \sum_{i=1}^{m} a^3_i$$

is the square of a rational number, and $S(N) = 1$.

Here's an interesting problem from Douglas J. Hoylman:

3 Pascal's triangle can also be written in rectangular form, in which case it looks like this:

1	1	1	1	1	...
1	2	3	4	5	...
1	3	6	10	15	...
1	4	10	20	35	...
⋮	⋮	⋮	⋮	⋮	

The first row and first column consist entirely of 1's and the other entries are found by adding the number to the left and the number above. If this array is continued to n rows and n columns, where n is any positive integer, prove that the determinant of the resulting matrix is 1.

Here is some magic from David DeWan:

4 A magic square is a square matrix of numbers such that rows, columns, and diagonals all sum to the same total. Create a 5 x 5 magic square using 25 two-digit numbers composed of the digits 0, 1, 6, 8, and 9. The magic square must also work when turned upside-down (90 becomes 06, etc.).

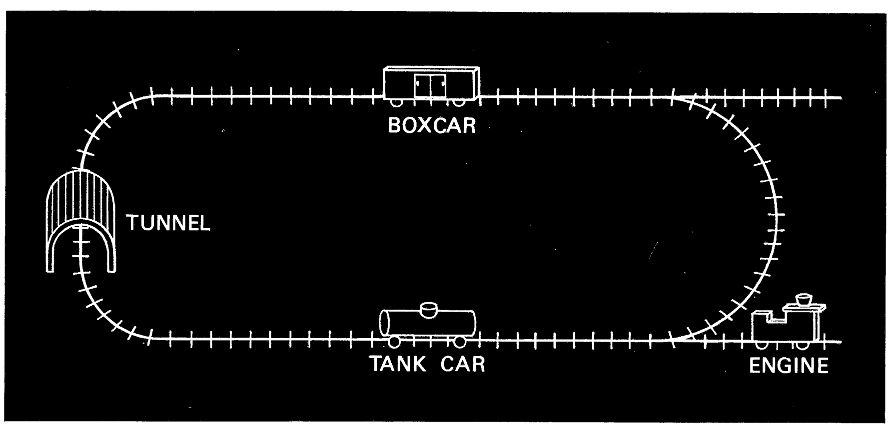
Smith D. Turner writes:

5 Determine a rational number whose square, when increased or decreased by 5, is still a square.

Speed Department

Donald F. Morrison figures the Penn Central needs some help:

SD1 On the railroad below, the tank car and boxcar can be pushed or pulled, singly or in tandem, but cannot move on their own. Only the engine can pass through the tunnel; the other two cars are too large. The problem is to interchange the tank car and boxcar and end up with the engine on the same siding it started on.



Frank Rubin proposes the following:

SD2 Arrange three points within a unit square so as to maximize the minimum between two of them.

Solutions

This month's solutions are to problems published in the May, 1970, issue of *Technology Review.*

31 The number 1,729 is an "interesting" number because it is the first number which is the sum of two positive cubes. Solve the corresponding problem for squares, fourth powers, and fifth powers.

Judith Q. Longyear writes:
Presumably what is wanted is the first number which can be written in two *different* ways as the sum of k powers; otherwise $2 = 1^k + 1^k$ solves all of them. If either of the proposers has the solution for k = 5, any journal on number theory or diophantine analysis will enjoy publishing it.

$1^1 + 3^1 = 2^1 + 2^1 = 4$

$1^2 + 7^2 = 5^2 + 5^2 = 50$

$1^3 + 12^3 = 9^3 + 10^3 = 1{,}729$

$134^4 + 133^4 = 158^4 + 59^4 = 635{,}318{,}657$

William Ackerman, Michael Rolfe, R. Robinson Rowe, and Samuel S. Wagstaff, Jr., the proposer, also responded.

32 In a league of 2n teams, each team plays every other team exactly once during a season. What is the greatest possible number of teams that can have a winning season? (Assume no ties.)

The following is from Robert Lack:
The answer is 2n − 1 teams. In a league of 2n teams, each team would play 2n − 1 games (under the conditions specified). In order for a maximum number of teams to have a winning record, 2n − 1 teams would have records of n games won and n − 1 games lost, while one team would have a record of no games won and 2n − 1 games lost. This works out in all cases because if 2n − 1 teams have each +1 in the won column (one more game won than lost), this equals +2n − 1 in the won column total. To balance this, one team has +2n − 1 in the *lost* column and 0 in the won column. (The total games won and lost for all 2n teams must be equal.) To demonstrate this, let n = 3 or six teams. At best, 5—or 2n − 1—teams can have a winning record:

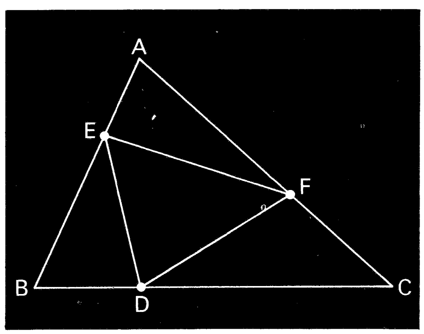
Won	Lost		Won	Lost
3	2	or	n	n − 1
3	2		n	n − 1
3	2		n	n − 1
3	2		n	n − 1
3	2		n	n − 1
0	5		0	2n − 1
15	15		5n	7n − 6

Since − 6 = − 2n, 7n − 6 = 5n. Therefore, in a league of 2n teams (under the conditions specified) a maximum of 2n − 1 teams can have a winning season.

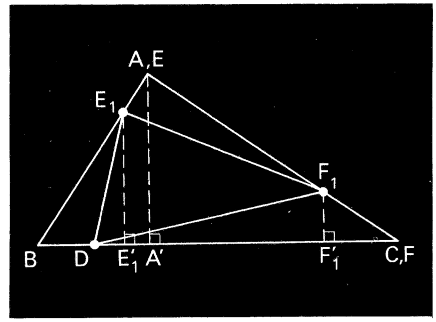
Also solved by William Ackerman, Daniel S. Diamond, James W. Dodson, Donald Forman, Winslow H. Hartford, Leon M. Kaatz, Judith Q. Longyear, Donald F. Morrison, John E. Prussing, R. Robinson Rowe, Frank Rubin, Les Servi, and the proposer, Douglas J. Hoylman.

33 Given any triangle ABC and a point D on segment BC, find (without using calculus) points E on AC and F on AB such that triangle DEF has maximum area.

I hate to keep appealing to Frank Rubin, but all other responses were either incomplete or nearly illegible. Here is Mr. Rubin's:



Answer: If BD ≦ DC, then choose E = A, F = C; if BD ≧ DC, then choose E = B, F = A. (Note that when BD = DC, there are two equally good choices). For proof, we will consider only the case where BD ≦ DC. Then the area of △DEF = AA′·DC. Suppose we chose some other $\triangle DE_1F_1$, then if $F_1F_1' \leqq E_1E_1'$, $A(DE_1F_1) \leqq A(DE_1F) = E_1E_1' \cdot DC \leqq AA' \cdot DC = A(DEF)$, with equality maintained only when $E_1 = E$ and $F_1 = F$. On the other hand, if $F_1F_1' > E_1E_1'$, then we have by the same argument $A(DE_1F_1) \leqq A(DBF_1) \leqq A(DBA) < A(DEF)$. This proves that the choice E = A and F = C is maximum when BD ≦ DC, and the argument for BD ≧ DC is the same.



William Ackerman, John E. Prussing, and Michael Rolle also responded.

34 A census taker stops at a house, notes down the number on the door, and knocks. When a woman answers, he asks her age and notes the answer. Then he asks if anyone else lives at the house; she replies that three other people live there. Upon asking their ages, he is given the reply that the sum of their ages equals the number on the door and their product equals 1,296. He does some quick computation and then asks if the oldest of the three is older than the woman to whom he is talking. She replies that the oldest is younger than she. What are the ages of the three? What is the house number?

James P. Friend notes:
Of all the possible triple factorizations of 1,296, only two have the same sum: 2, 8, 81 and 1, 18, 72. The census taker knew the house number was 91, and he knew the above factorizations. Obviously the woman answering the questions was between 72 and 81 years of age (assuming an 18-year-old does not qualify as a woman). Since the oldest of the three other residents was younger than the woman answering the question, the ages of the three residents are 1, 18, and 72. (The assumption of an 18-year-old not

being a woman may be dubious if she is the mother of the one-year-old.)

Also solved by High C. Barrett, Richard S. Galik, Woodrow M. Hazel, R. Robinson Rowe, and Frank Rubin.

35 Given the following hand, with the bidding as indicated, show how the declarer can take 11 tricks, assuming the diamond finesse must be successful.

♠ 8 4 2
♥ A 6 5
♦ Q J 9 8
♣ K 8 7

♠ A Q 9 6 3 — ♠ J 10
♥ K Q J 10 8 7 — ♥ 9 3 2
♦ — — ♦ K 7 6 5 4 3 2
♣ 10 3 — ♣ 9

♠ K 7 5
♥ 4
♦ A 10
♣ A Q J 6 5 4 2

The bidding: South—one club, West—double, North—redouble, East—one diamond; South—three clubs, West—four hearts, North five clubs, East—pass. West opens with ♠K.

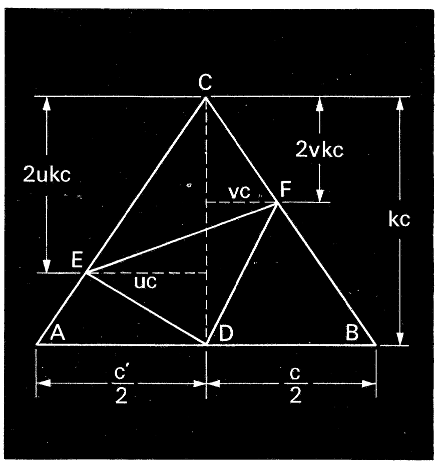
Rex Ingraham solved all the problems involved—including some that were not intended; he proposes three of the latter: What card did West lead, really? Who goofed, and why? Why did Warren Himmelberger, the proposer, guarantee the diamond finesse?
In last-things-first-order: If Warren Himmelberger didn't mean to cue the solution he must have meant to hide it; either way I'd only fault him for the mention of the finesse because a bridge player would have to risk it on his own, anyway. It must have been a proofreader who goofed—not a bridge fan—because Allan Gottlieb certainly knows ♠K ≠ ♥K and Warren Himmelberger knows ♠K is not among West's assets. West must have opened ♥K, because of all his cards this is the only opening to present a problem which can be solved without depending on a pure guess by the declarer.
A complete solution and adequate explanation of the reasoning goes something like this, I think: West opens ♥K. The declarer considers the old common-law "Who looks ere leaps may live to leap another day" and the ancient adage, "Aces ain't always assets," sees that he will win no spade trick without a helpful lead from West, and concludes that he can well afford to duck the first trick and does so. West now has no lead which will not give the declarer 11 tricks; he has already blown the defense, although he cannot know this. If he buys the temptation, to lead a low spade (East's ♠K and diamond return could set the declarer down three or four tricks), the declarer's ♠K will actually score him an overtrick. If West opts to fill his book while sure of ♠A, then any lead to the third trick will put the declarer in to score 11 tricks. Whether at the second or third trick, the declarer can win in his own hand (the dummy would again duck any heart continuation), complete drawing trumps to the dummy's ♣K, and play ♦Q. East's temptation to cover in desperate hope to lead a space through the declarer's weakness could prevail; if so, the declarer's ♦A will win and at once give him 11 or 12 established tricks. But when East ducks the ♦Q, so also must the declarer duck—not because Warren Himmelberger guaranteed the finesse but because its success is vital to fulfillment of the contract and practically assured by East's bid and West's repeated failure to lead any diamond. When ♦Q holds, ♦A is suddenly changed from asset to liability. But the declarer can purify his hand by stashing it beneath dummy's ♥A—and does so. Now the declarer can continue dummy's diamonds to ruff away East's ♦K, return to the dummy's ♣8, to cash any remaining diamonds, and claim his remaining trumps for 11 or 12 tricks, as the case may be.

Also solved by James W. Dodson, James P. Friend, Donald Forman, Cmdr. R. H. Gaunt, Winslow H. Hartford, Leon M. Katz, T. C. Robinson, Patrick J. Sullivan, Edmund J. Thimme, Alan B. Wright, and the proposer, Warren Himmelberger.

Better Late Than Never

13 Find conditions on the ratio of the altitude to the base of isosceles triangle ABC such that the inscribed triangle DEF with maximum area (D is at the midpoint of AB) has FE parallel to AB.

Charles S. Rall notes that the solution as published in the April, 1970, issue is incorrect. Indeed, he says, as the problem is worded there is no solution:



As was done in the published solution, let the area of DEF = A and the area of ABC = A′. In addition, let f = (u + v − 4uv) and

$$\underline{x} = \begin{Bmatrix} u \\ v \end{Bmatrix}$$

so that the matrix notation may be used. As published in the solution, one has $A' = \frac{1}{2}kc^2$ and $A = (u + v - 4uv)$
$A' = fA'$. One should note here that because $A/A' = f$ is not a function of k, the ratio of the altitude to the base cannot affect the answer, as could be seen from the published solution. Continuing,

$$\left[\frac{\partial}{\partial \underline{x}}\left(\frac{A}{A'}\right)\right]^T = \left(\frac{\partial}{\partial \underline{x}} f\right)^T = \begin{Bmatrix} 1 - 4v \\ 1 - 4u \end{Bmatrix}.$$

Equating this quantity with zero does indeed give a stationary value at

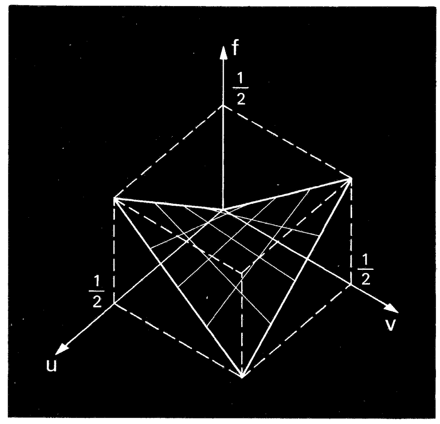
$$\underline{x} = \begin{Bmatrix} 1/4 \\ 1/4 \end{Bmatrix} = \begin{Bmatrix} u \\ v \end{Bmatrix}.$$

However, looking at the matrix of second derivatives,

$$\frac{\partial^2 f}{\partial x^2} = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix},$$

one sees that this stationary point published in the solution as a minimum is instead a saddle point. Let us look at the value of the function f over the range of permissible values for u and v,
$u, v \in [0, \frac{1}{2}]$.
The permissible interval for u and v is assumed to be closed so that a maximum actually does exist. The sketch below of the value of f versus u and v in the permissible region demonstrates that there are two maxima,

$$\underline{x} = \begin{Bmatrix} 0 \\ 1/2 \end{Bmatrix} \text{ and } \underline{x} = \begin{Bmatrix} 1/2 \\ 0 \end{Bmatrix}.$$



In other words, the inscribed triangle with maximum area and with D the midpoint of AB occupies either the left half or the right half of triangle ABC and has an area of exactly one-half the area of the larger triangle. FE is not parallel to AB but is coincident with either AC or BC.

14 Find a function f defined on the entire real line such that
1. f is bounded and strictly increasing;
2. f is continuous at each point x; and
3. $\lim_{x\to-\infty} f'(x) \neq 0 \neq \lim_{x\to\infty} f'(x)$.

Two mistakes in a row; William Ackerman corrects me, writing that there is no function f defined on the reals as specified. His proof:
Since $\lim_{x\to\infty} f'(x) \neq 0$ and f is increasing, $f'(x) \geq 0$ everywhere, and VS, $\exists X$ such that
$x > X \Rightarrow |f'(x) - \lim_{y\to\infty} f'(y)| < \delta$
$\lim_{y\to\infty} f'(y)$ must be greater than 0, so, letting
$\delta = [\lim_{y\to\infty} f'(y)]/2$,
$\exists X \ni x > X \Rightarrow f'(x) > [\lim_{y\to\infty} f'(y)]/2$,
so f (x) is bounded away from zero for $x > X$, and so f(x) grows faster than a first-degree polynomial. Specifically, by the Mean Value Theorem, $x > X =$

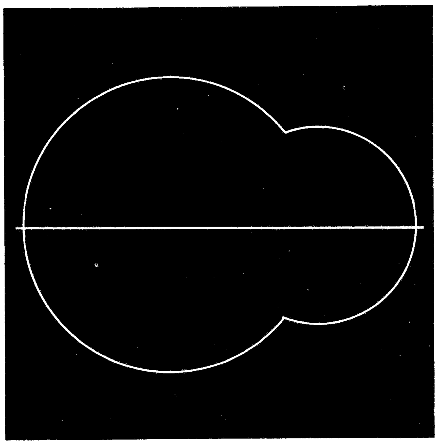
$f(x) - f(X) = f'(\xi)(x - X)$ for some $\xi \ni (X, x)$, so
$f(x) - f(X) > [\lim_{y\to\infty} f'(y)]/2(x - X)$.
Since f is bounded, $f(x) \leq D \; \forall x$, but if

$$x = \frac{[D - f(x)]}{[\lim_{y\to\infty} f'(y)]/2} + X$$

then
$D - f(X) = [\lim_{y\to\infty} f'(y)]/2(x - X)$.
Therefore $f(x) - f(X) > D - f(X)$, so $f(x) > D$.
The published solution correctly satisfied properties 1 and 2 ($\tan^{-1}$, tank, and error function are other examples), but the attempt to add a discontinuity to f' at infinity failed because:
1. $f'(\infty)$ is not defined;
2. Even if it were, adding a discontinuity there would not solve the problem. The problem concerns $\lim_{x\to\infty} f'(x)$, and hence constrains f' near ∞, not at ∞.
It does not matter whether the limit does or does not equal $f'(\infty)$, as long as it does not equal zero.
3. The attempt to add the discontinuity at 1 and shift 1 to ∞ by mapping (0, 1) to $(0, \infty)$ would not work in any case because $1 \notin (0, 1)$ and $\infty \notin (0, \infty)$.

16 Find a curve having nonconstant radius of curvature such that all the centers of curvature lie on the x axis.

Donald E. Savage writes that R. Robinson Rowe's solution as published in May, 1970, is in error:
First, as a minor point, equation 3 is in error (by a minus sign) as one can see from either of his diagrams. For example, along the arc OU, $y > 0$, $y' > 0$, and yet $y'' < 0$. But my major criticism is of his equation 4. Differentiating both sides, I get
$y'' = -2[\sqrt{a^4 - y^4}]/y^3 + [y/\sqrt{a^4 - y^4}]y'$.
Substituting for y' and then multiplying by y, I get
$yy'' = -2a^4/y^4$.
But $1 + y'^2 = a^4/y^4 \neq yy''$, as equation 3 requires.
Having criticized his solution, I will now make my own available for criticism:
Part I: "Nice" functions. From Burington's Tables, the y coordinate of the center of curvature corresponding to the point x,y on $y = f(x)$ is given by
$k = y + (1 + y'^2)/y''$.
Setting this equal to zero, I get
$yy'' = -(1 + y'^2)$.
To solve this, let $p \equiv y'$, so that
$y'' = dp/dx = dp/dy \cdot dy/dx = p(dp/dy)$.
Substituting this in the above equation, and letting $u \equiv p^2$,
$yy'' = yp(dp/dy) = y/2(du/dy) = -(1 + y'^2) = -(1 + p^2) = -(1 + u)$.
Therefore $du/(1 + u) = -2dy/y$.
Integration gives
$u + 1 = p^2 + 1 = (y/a)^{-2}$,
where a is an integration constant.
Therefore
$p = dy/dx = (\sqrt{a^2 - y^2})/y$.
Integrating again,
$-\sqrt{a^2 - y^2} = x - x_0$,
where x_0 is the second integration constant. Thus $(x - x_0)^2 + y^2 = a^2$, which is the equation of a circle of radius a and center at x_0, 0. Thus for "nice" functions (having continuous second derivatives) it appears that there are no curves of nonconstant radius having all centers of curvature on the x axis.
Part II: "Goshawful" functions. A facetious answer to problem 16 is, "That's easy; two circles of different radii with centers on the x axis." To get around the possible objection that this is really two curves, one can put the circles together and erase certain parts:



Pursuing this line of reasoning *ad infinitum* (or, perhaps, *ad nauseam*), the answer to problem 16 can be shown to be "any ol' continuous curve of nonconstant radius." To show this, note that any ol' continuous curve can be approximated by another one obtained from the first by (1) marking off the first into small segments, (2) approximating each segment with a circular arc whose ends lie on the ends of the segment and whose center of curvature lies on the x axis. Then by making the segments smaller without limit, the approximating curve comes arbitrarily close to the original curve, all the while having the required properties—almost everywhere. (Do I hear you muttering something about the "measure" of the points where it doesn't?)

Responses were also received from Donald Forman and Michael Rolle.

20 A said to the farmer, "I know you own a rectangular plot in that 20-by-20 section, and I know the area of your plot. Is the length greater than twice the width?" B said to the farmer, "Before you answer let me state that I knew the width, and I now know the length." C said, "I did not know the length, width, or area, but I now know the dimensions."
What are they?

The proposer, John Mandl, disagrees with the solution published in the May issue. He writes:
The solution stated that $L_{max} = 20$. This is incorrect. The upper limit of L is $20\sqrt{2}$, since the rectangular plot could lie along the diagonal of the 20-by-20 section. The solution stated that $W \leq 10$, and yet the final statement is that $L = W = 10\sqrt{2}$; these statements are contradictory.
This approach to the problem is one I had not considered; my original approach was one which confined the analysis to the case of integer length and width.
This approach resulted in the following:
1. Of all possible values for the area, we can immediately discard those areas bounded by unique L and W. For instance, if A knew the area to be 7, the dimensions would have to be 7 x 1, and there would have been no need for his question.
2. A's question, "Is the length greater than twice the width?" was designed so that either a Yes or No answer would enable him to determine the dimensions. The question itself eliminates several possible areas—namely, those areas which show up twice on either side of the line defined by $L > 2W$ (such as A = 18, which can be formed by 18 x 1, 9 x 2, or 6 x 3, because a possible Yes answer would not have told A whether the dimensions were 18 x 1 or 9 x 2).
3. After eliminating all the unique and the duplicated areas, B—knowing the W—had only one possible area left and consequently could determine L.
4. C examined all the widths for the remaining areas, found only one W which was accompanied by a single area, and was able to duplicate B's analysis.
The final solution with this approach is W = 11, L = 12. (Incidentally, plots of 27 x 1, 26 x 2, etc., will fit in the 20-by-20 section.)

Responses also received from M.B. Brilliant, Robert C. Fleetham, and T. A. Ginsburg.

There have been a number of additional responses to other earlier problems:
12 John Price
18 James W. Dodson
21 Frank Rubin
22 Frank Rubin
23 Donald F. Morrison
25 Frank Rubin
27 James W. Dodson, Robert Pogoff, and Robert C. Hall
30 William Ackerman, R. N. Assaly, Gerbert Barnard, Harold Donnelly, Woodrow M. Hazel, Charles Heiberg, Thomas H. Kick, J. A. Jacobs, J. T. O'Connor, G. Stephen Pittman, Victor W. Sauer, Balbir Singe, and—of course—Frank Rubin.

Allan J. Gottlieb, who studied mathematics at M.I.T. in the Class of 1967, is a member of the Mathematics Department at Brandeis University, Waltham, Mass. 02154. Send problems and solutions to him at that address.