The Canary in the Submarine, etc.

For part of the summer my residence is changed to the "real world"—i.e., I get a job. It is my personal belief that teaching and learning mathematics is also work, but this is not the majority opinion. I must admit, however, that rigid deadlines do seem a little foreign. If anyone can discover a method of having each sunset separated by 42 hours, I will gladly give him a lifetime subscription to *Technology Review.*

Enough of my problems. Here are some of yours.

Problems

Benjamin Whang offers the following "delicate" problem:

41 A canary is hovering inside a submarine when the submarine finds its neutral equilibrium in water. When the canary gently lands on the submarine deck, does the submarine go down?

42 Andrew D. Egendorf and George Starkschall (both were my classmates at M.I.T.) want you to find the next term in the following series:
18 46 94 63 52 —
(You should not forget that Egendorf is known to be "zonked out.")

Through Fred W. Goldthwait, Secretary of M.I.T.'s Class of 1905, we have the following from Robert S. Beard, a classmate:

43 Outline the geometrical method of drawing the five-star insignia of our top military commanders. If W is the width of any star, can anyone determine the ratio K of the distance between adjacent star points to W?

The following is from David L. Arenberg:

44 A certain physicist is studying a nuclear reaction with very precise equipment. He can sort out and store separate atoms and observe when a gamma ray is emitted by any of his collection, but he cannot tell which one has changed or tell whether a neutron was released or absorbed also. His equipment has a balance sensitive enough to weigh any or all of the atoms and finally to sort out the single atom and determine the process. After 282 atoms are isolated, a gamma ray is detected; as the weighing process is exceedingly tedious, the physicist seeks to determine the method that will yield the answer in the least number of weighings. While the physicist is checking the data on a computer—with a tremendous amount of identical parallel circuits feeding in—there is a breakdown; the operators have difficulty in isolating defective circuits. The physicist therefore comments, "If you people have as sensitive a nulling ammeter as I have a balance and can parallel all the inputs and outputs as you please, I can determine which component is open or shorted in less than 11 comparisons if the total number of elements is not much more than 68,000." To fulfill his promise, what (1) was the least number of weighings of the atoms made by the physicist; and what (2) is the greatest number of circuits he could check with 11 comparisons?

The last new problem is from Lydall D. Morrill, Jr.:

45 "Here's a problem for you, Peter," said his friend Barry: "What time is it when the spread between a clock's hands (measured the short way) is an integral multiple of 13 minutes? To keep it simple, I'll tell you that no fractional minutes are involved." Peter replied instantly, "The trivial solution is 12:00, of course—assuming you count zero as an integral multiple." "No," Barry said, "the hands are pointing in different directions. Try again." After a few minutes' thought, Peter said, "I need a hint: Is it before or after 4:00?" "Telling you that would give away the solution," Barry smiled. Shortly Peter announced the answer. Given the facts above, the solution is unique. What is it?

Speed Department

John Reed submits the following two-part speed problem:

SD18 What are the keys to the following two infinite series:
(A): 2, 3, 4, 6, 8, 12, 14, 18, 20, 24, 30, 32, . . .
(B): O, T, T, F, F, S, S, E, N, T, E, T, T, F, F, . . .

Solutions

24 Construct a triangle given the three altitudes.

Apparently *Technology Review* gets around. The following is from Rüdiger Dierstein and comes to us "mit luftpost" from Germany:

Let a, b, and c be the sides of a triangle and h_a, h_b, and h_c the corresponding altitudes. Then it holds:
$a : b = h_b : h_a$
and
$b : c = b_c : h_b$;
thus
$a : b : c = h_b : h_a : (h_a h_b)/h_c$.
Since the relation $h_a : h_b : h_c$ is known, we may construct a triangle with sides
$\bar{a} : \bar{b} : \bar{c} = h_b : h_a : x$
where x may be constructed from
$x = (h_a h_b)/h_c$ or $x : h_a = h_b : h_c$.
Using triangle $\bar{a}\,\bar{b}\,\bar{c}$ it is no problem to get the desired triangle a b c by applying a similarity transformation using one of the given altitudes h_a, h_b, or h_c.

Also solved by Captain John Woolston, James J. Heyman, and R. Robinson Rowe.

				G	L	O	B		
				L	O	V	E		
F	L	A	B	O	V	A	L		
L	I	R	A	B	E	L	T		
A	R	A	B	A	R	E	N	D	S
B	A	B	E	R	I	N	E	A	T
		G	L	E	N	D	A	T	A
		L	Y	R	E	S	T	A	Y
		E	R	G	S				
		N	E	S	T				

25 Complete the following (the unshaded area) given the shaded area so that all squares contain only words and the word in row i of a square is the same as the word in column i of that square.

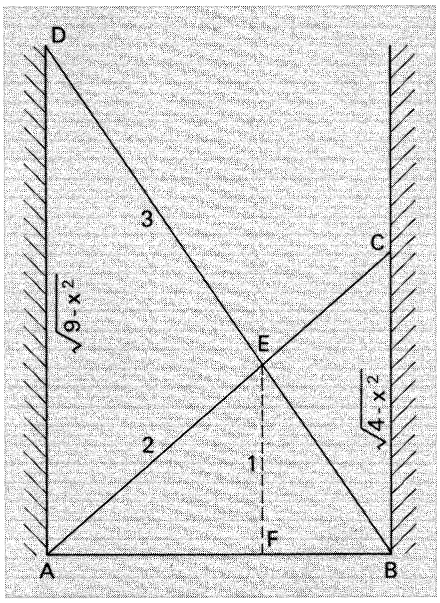
Several different solutions were received. The one chosen (above) was a joint effort of Bernice Joy Blumenreich and Michael S. Bodner. Others came from Winslow H. Hartford, Captain John Woolston, R. Robinson Rowe, David Blank, Susan Bock, and the proposer, Robert S. Cox.

26 Show that there are infinitely many integral solutions to
$x^3 + y^3 + z^3 + w^3 = 0$.

Here is a composite solution obtained by taking the best parts of several replies: Obviously $y = -x$, $z = -y$ give trivial solutions. Also, multiplying any solution by any cube yields another solution, so we assume no common factors. One solution is
$3^3 + 4^3 + 5^3 - 6^3 = 0$,
another is
$8^3 + 6^3 + 1^3 - 9^3 = 0$,
and a third is
$10^3 + 9^3 - 12^3 - 1^3 = 0$.
Infinitely many primitive solutions come from
$(n^3 + 1)^3 + n^3(n^3 - 2)^3 + (2n^3 - 1)^3 = n^3(n^3 + 1)$,
but this is hard to see (except to Euler, 1756). Even worse is the identity:
$[m(m^3 + 2n^3)]^3 + [m(n^3 - m^3)]^3 + [n(n^3 - m^3)]^3 = [n(2m^3 + n^3)]^3$.

This solution is from George E. Keith, Jr., Douglas J. Hoylman, Michael Krashinsky, R. Robinson Rowe, Russell L. Mallett, and John E. Prussing.

27 A 60-ft. ladder and a 40-ft. ladder intersect 20 ft. above street level. How wide is the street?



R. Robinson Rowe submits the following: There is a simple geometric relation between the heights of intercepts on the walls and intersection of ladders, like the lens-focus formula:
$1/AD + 1/BC = 1/EF$.
In this case, the ladders are multiples of EF, so it will be convenient to use EF as a unit, with $AC = 2$, and $BD = 3$. Then, letting $AB = x$, the formula derives the relation
$\sqrt{9 - x^2} + \sqrt{4 - x^2} = \sqrt{9 - x^2}\sqrt{4 - x^2}$.
Removing the radicals by squaring twice derives the octic
$x^8 - 22x^6 + 163x^4 - 454x^2 + 385 = 0$,
which I solved as a quartic in x^2 by Horner's method, then by square root. Finally, since the unit was the EF, where 1 EF = 20 ft., the width of the street was 20x, or 24.623 714 48 ft.

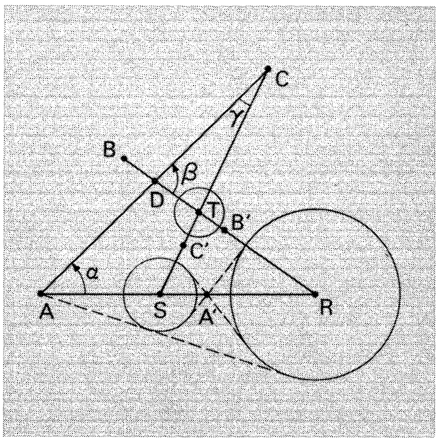
Also solved by John E. Prussing, Russell L. Mallett, John Reed, George Van Arsdale, Lawrence M. Kieran, John D. Fogarty, Major F. H. Cleveland, William McClary, Frank G. Satkiewicz, W. Everett Swift, Clark Thompson, Arthur W. Anderson, Roy G. Sinclair, and George E. Keith, Jr.

28 Solve the following two "concealment ciphers":
Pediatric researchers find that apparent learned errors attenuate reliable actions, channelling unavoidable patterns at neural circuitry. Assuming aggressive or regressive patterns at an early age tends to reinforce the unreliability.
In legal disputes, the most rational attempts to respond to accusations necessitate thorough research, high standards, astute observation, the psychologist's feeling for allusion, and all possible care to avoid countersuits.

Douglas J. Hoylman has the following guesses:
"The first looks like 'spare the rod and spoil the child,' and would the second by any chance be 'look before you leap'?"

Also solved by R. Robinson Rowe.

29 Show that given three unequal circles whose centers are noncolinear, the points of intersection (A, B, and C) of the three pairs of common tangents are colinear.



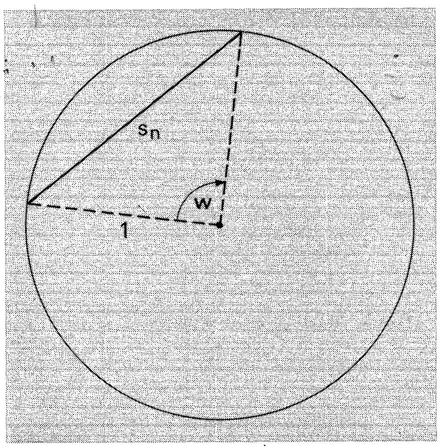
Russell L. Mallett's solution is:
Let R, S, and T be the circle centers and r, s, and t the corresponding radii, with $r > s > t$. There are many more points of intersection than stated, unless the circles themselves intersect. Nevertheless, there are always three points of intersection A, B, and C such that ASR, BTR, and CTS are straight lines. Simple geometry gives:
$AR/AS \cdot CS/CT \cdot BT/BR = r/s \cdot s/r \cdot r/t = 1$.
The figure shows A, B, and C not colinear. Let AC and BR intersect at D. Then the law of sines gives:
$DR/AR \cdot CT/DT \cdot AS/CS = (\sin\alpha)/(\sin\beta) \cdot (\sin\beta)/(\sin\gamma) \cdot (\sin\gamma)/(\sin\alpha) = 1$.
Combining these two results gives
$BT/BR = DT/DR \rightarrow BT = DT$,
so B must coincide with D and thus A, B, and C are colinear.
When the circles do not intersect, there are also pairs of common tangents which intersect in points A', B', and C', falling between circle centers. Each pair of these "interior" intersection points is colinear with A, B, or C.

Also solved by George E. Keith, Jr., Major F. H. Cleveland, Ted Leahy, Roy G. Sinclair, Douglas J. Hoylman, and R. Robinson Rowe.

30 Given the following construction, prove or disprove that the resulting figure is a regular pentagon:
1. Draw a circle with center at 0.
2. Draw line CD through the center of the circle.
3. Construct the perpendicular bisector to CD, line AB.
4. Construct the perpendicular bisector to OD, dividing it into two equal parts, OE and ED.
5. Place the compass point on E and the lead on A, and draw arc AF.
6. Place the compass point on A and the lead on F, and draw arc GH.
7. Leaving the compass with this setting, place its point on G and locate I on the circle.
8. Leaving the compass with this setting, place its point on H and locate J on the circle.
9. Draw the pentagon using points I, G, A, H, and J.
Trigonometric analysis of the problem shows that it reduces to demonstrating that $\sin 36° = (\sqrt{10 - 2\sqrt{5}})/4$. Can this be done?

The definitive work on this problem comes from Howard A. Robinson, Chairman of the Department of Physics at Adelphi University, whose absorbing letter follows in its entirety:
The solution to this problem is well known and given in Bruckner's famous book, *Vielecke und Vielflach* (Leipzig, 1900). (On second thought I'm not so sure how famous the book actually is. I came across it by chance years ago in the New York Public Library from whence it shortly thereafter disappeared. Repeated search found it in no other library our librarian could uncover. However, it has recently turned up again in some other corner of that same library and I have had a copy made for Adelphi University. It is an invaluable compendium for the

solution of almost any problem in polygon or polyhedral geometry, and I would suggest that the Brandeis and M.I.T. Libraries should also have it).
The general solution for a polygon of any number of sides follows:
Consider a circle of unit radius with an n-sided regular polygon inscribed in it.



Let the length of a single side be s_n and the included angle be w. Then
$\sin (nw/2) = 0.$
But $\sin (nw/2)$ can be expanded into the following series:
When n is odd:
$\sin (nw/2) = n \sin w/2 - [n(n^2 - 1^2)/3!] \sin^3 w/2 - n[(n^2 - 1^2)(n^2 - 3^2)]/5! \sin^5 w/2 \pm \ldots$
When n is even:
$\sin (nw/2) = \{n \sin w/2 - [n(n^2 - 2^2)3!] \sin^3 w/2 + n[(n^2 - 1^2)(n^2 - 4^3)]/5! \sin^5 w/2 \pm \ldots\} \cos w/2.$
Thus if $n = 5$, $w = 72°$, $\sin w/2 = s_5/2$ (for a circle of unit radius) and
$0 = s_5^4 - 5s_5^2 + 5,$
from which
$s_5 = \sqrt{5}/2 \pm \sqrt{5}$ and
$\sin 36° = (\sqrt{10 \pm 2\sqrt{5}})/4$, as requested.
The meaning of the + sign under the radical is of interest. In the theory of polygons the concept of polygon type *(Art espèce)* was early introduced. Thus if the circle be divided into n equal parts, each point can be joined to its next ($a = 1$) to form a polygon of the first type; it can likewise be joined to its next but one ($a = 2$) to form a polygon of the second type. In the case of the pentagon, ($a = 3$) and ($a = 4$) lead to the same two types of polygon as in the case when $a = 1$ and $a = 2$. The plus sign gives the length of side of a pentagon of type two. (The nomenclature is due to Poinsot).

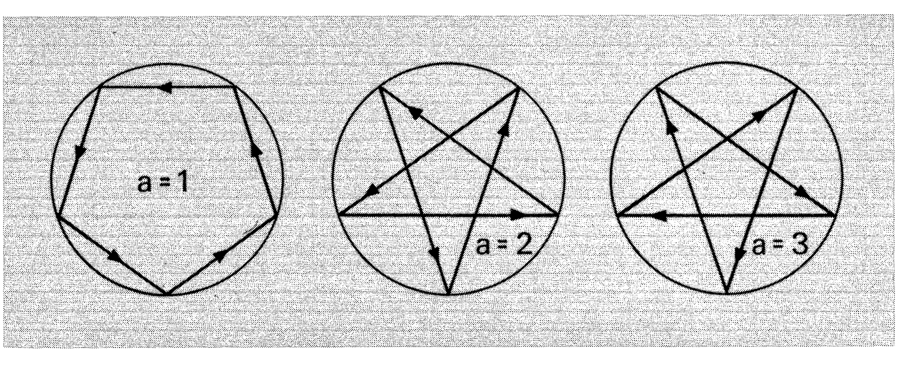
Similarly for $n = 10$ there will be four types ($a = 1,2,3,4$) and from the expansion $s_{10}^4 - 8s_{10}^6 + 21s_{10}^4 - 20s_{10}^2 + 5 = 0.$
This can be factored into
$(s_{10}^4 - 5s_{10}^2 + 5)(s_{10}^4 - 3s_{10}^2 + 1) = 0.$
The roots of the first factor are identical with those of the pentagon and correspond to the lengths of the 10-sided figure of the second and fourth types. The second factor gives roots corresponding to the edge length of the first and third type, that is
$s_{10} = \sqrt{(3 \pm \sqrt{5})/2} = (\sqrt{5} \pm 1)/2.$
The original problem raised by Mr. Megill forms part of a wider problem known as the Kreisteilung. The problem of dividing the circumference of a circle into n equal parts using only a ruler and compass is very old, and it was long known that such a construction was possible for the cases $n = 2^h$, 3 and 5 or any combination of these cases. Gauss showed that the construction is possible for every prime number n of the form
$n = 2^{2^u} + n$
but that it was impossible for all other prime numbers and prime number powers since in these cases all solutions involved equations higher than second order, the the roots of which are not constructable with ruler and compass. The cases $u = 0$ and $u = 1$ lead to the cases of $n = 3$ and $n = 5$ above. For $u = 2$, $n = 17$, the construction which Gauss carried out himself and which has been widely discussed. For $u = 3$, $n = 257$ and for $u = 4$, $n = 65{,}537$. Since both n are prime the constructions are realizable. The case of $n = 257$ was constructed by Richelot in 1832 (*Crelles Journal* Vol. 9) and the case of $n = 65{,}537$ construction was carried out by Hermes after a 10-year labor, the results of which were deposited in the collections of the Mathematical Seminar in Göttingen. $u = 5,6,7$ do not lead to prime numbers; hence the solutions do not exist. No one (at least up to 1900) seems to have investigated the case of $u = 8$. Some of your readers may wish to pursue this matter further, although some suspicion seems to have arisen that $u = 4$ is the last realizable case. Would any reader care to evaluate Mr. Hermes?

Also solved by Matthew J. Relis, David B. Smith, Paul Guilden, John E. Wieschel, James Marler, Jr., Zaul Hasan, John L. Sampson, Jan M. Chaiken, Richard Lipes, John L. Maulbetsch, James R. Schueler, Thomas Tredici, Gilbert Shen, Daniel E. Jones, John W. Goppelt, Winslow H. Hartford, J. J. Cerullo, Howard S. Jarrett, C. Peter Lawes, D. Thomas Terwilliger, Robert D. Larrabee, F. R. Morgan, Major F. H. Cleveland, Robert A. Schumaker, William H. Peirce, Irving L. Hopkins, Edward S. Eby and Murray B. Sachs (jointly), Ruth Helfish and Timothy M. Barrows (jointly), Stephen Scheinberg, W. Allen Smith, Thomas W. Summers, Joel L. Ekstrom, William J. Wagner, Arthur A. Hauser, Jr., Norman C. Peterson, Roy G. Sinclair, Ted Leahy, George E. Keith, Jr., Russell L. Mallett, Arthur W. Anderson, Michael Krashinsky, and R. Robinson Rowe.

Better Late Than Never

Jerry Blum has responded on problems 11 and 14.

Julian Pathe attacks the published solution to problem 20. The original problem read, "A said to the farmer, 'I know you own a rectangular plot in that 20-by-20 section, and I know the area of your plot. Is the length greater than twice the width?' B said to the farmer, 'Before you answer let me state that I know the width, and I now know the length.' C said, 'I did not know the length, width, or area, but now I know the dimensions.' What are they?" Mr. Pathe writes:
I am distressed with the series of assumptions given for the solution in the May issue of *Technology Review.* They completely overlook the word *farmer,* thus failing to determine the units of measurement. These are rods and acres. B immediately knew the length when A said "20" (rods). C, "the remarkable mental gymnast" as stated in the February issue, knew that 8 x 20 = 160 square rods = 1 acre. The published answer $L = W = 10\sqrt{2}$ (microns?) is unsupportable. Is the future of our country safe in the hands of those who ignore facts and have never suffered for their eagerness to live with careless assumptions? I never saw the problem before the February issue and would like to know the "correct" answer.
I have enjoyed reading *some* of the problems. My 13-year-old son last year did the "census taker" in two hours, but the effort required a $2 bribe.



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