Allan J. Gottlieb

In Celebration of Spring

The most exciting event of the month in which this is written has been the arrival of spring. A local radio station, WBZ, threatened to cancel spring, and for a month it looked as though they weren't kidding.

I must issue another plea for speed problems. The supply is critically low.

Problems

31 The first problem is from Samuel S. Wagstaff, Jr., who recalls Leslie R. Axelrod's comment in the July/August, 1969, issue that 1,729 is an "interesting" number because "it is the first number which is the sum of two cubes." (But he notes parenthetically that Mr. Axelrod "should have taken greater care in stating that property of 1,729; he wants the number to be the sum of two *positive* cubes.") Mr. Wagstaff's problem: solve the corresponding problem for squares, fourth powers, and fifth powers.

The next offering is by Douglas J. Hoylman:

32 In a league of 2n teams, each team plays every other team exactly once during a season. What is the greatest possible number of teams that can have a winning season? (Assume no ties.)

The following is by Frank Rubin:

33 Given any triangle ABC and a point D on segment BC, find (without using calculus) points E on AC and F on AB such that triangle DEF has maximum area.

Russell A. Nahigian offers the following:

34 A census taker stops at a house, notes down the number on the door, and knocks. When a woman answers, he asks her age and notes the answer. Then he asks if anyone else lives at the house; she replies that three other people live there. Upon asking their ages he is given the reply that the sum of their ages equals the number on the door and their product equals 1,296. He does some quick computation and then asks if the oldest of the three is older than the woman he is talking to. She replies that the oldest of the three is younger than she. What were the ages of the three? What is the house number?

Warren Himmelberger submits the following bridge problem—supposedly, he says, a hand played in a public match half-a-dozen years ago.

35 Given the following hand, with the bidding as indicated, show how the declarer can take 11 tricks, assuming the diamond finesse must be successful.

	♠ 8 4 2 ♥ A 6 5 ♦ Q J 9 8 ♣ K 8 7	
♠ A Q 9 6 3 ♥ K Q J 10 8 7 ♦ — ♣ 10 3		♠ J 10 ♥ 9 3 2 ♦ K 7 6 5 4 3 2 ♣ 9
	♠ K 7 5 ♥ 4 ♦ A 10 ♣ A Q J 6 5 4 2	

The bidding started with South one club, West doubled, North redoubled, and East bid one diamond. South responded with three clubs, West four hearts, North five clubs, and East pass. West opens with ♠K.

Speed Department

The only contributor is John E. Prussing:

SD12 Show that the product of all primes less than 1,000 is an even number.

SD13 Here is a proof that all integers are odd:
Let P(n) denote the proposition that 1,2,3,...,n are all odd integers. The proof follows by induction:
1. P(1) obviously.
2. Assume P(k). If 1,2,...,k are all odd, then $k - 1$ is odd. By adding 2 to $k - 1$, one shows that $k + 1$ is also odd.
3. Thus P(k) implies P(k + 1) and the proof is complete.
What is the fallacy?

Solutions

14 Find a function f defined on the entire real line such that
1. f is bounded and strictly increasing;
2. f is continuous at each point x; and
3. $\lim_{x \to -\infty} f'(x) \neq 0 \neq \lim_{x \to \infty} f'(x)$.

Here is John E. Prussing's solution:
An example of such a function is the bounded, continuous, monotonic function defined by
$f(x) = 1 - e^{-x}$ for $x \geq 0$.
For $x < 0$, $f(x) = -f(-x)$.
Since for this function
$\lim_{x \to -\infty} f'(x) = 0 = \lim_{x \to \infty} f'(x)$,
we must also include a discontinuity in f' at $\pm\infty$. This discontinuity in f' will not affect the above-named properties. If having the discontinuity at infinity is unesthetic, one could map the interval $(0, \infty)$ into the interval (0, 1) by the transformation $y = x/(1 + x)$ and place the discontinuity at $y = 1$.

Also solved by John Pierce, Peter Ross, Mark Yu, Frank Rubin, R. Robinson Rowe, and Homer D. Schaaf.

15 This problem was a variation of one of last year's problems:
A mathematician moonlighting as a census-taker stops at his friend's house. In this census he is required to obtain the names and ages of all the occupants of the house. After writing down several names and ages the census-taker asks, "Are there any more people who live here?" His friend replies, "Yes, there are three more people that live here." When asked for their ages, the friend reports that the product of the ages is 1296 and the sum is the street number of his house. The census taker makes a few calculations and then says, "Just tell me one more thing: How many of the three are older than you are?" As soon as his friend replies, the census taker smiles, writes down the ages and leaves. What is the house number?
The variation proposes that two veterans (i.e., older than 18) discuss a similar situation where the house number is not known. One veteran asks how many of them are older. Which reply allows him to determine the house number?

Frank Rubin responds as follows:
This problem is undoubtedly the most unclearly stated problem you have ever

published. But to make a stab at it, I make the following assumptions:

1. The veterans are the census taker and his informant, not the three remaining occupants of the house.
2. The age of the informant is known to the census taken but not to us. We simply know that he is older than 18.

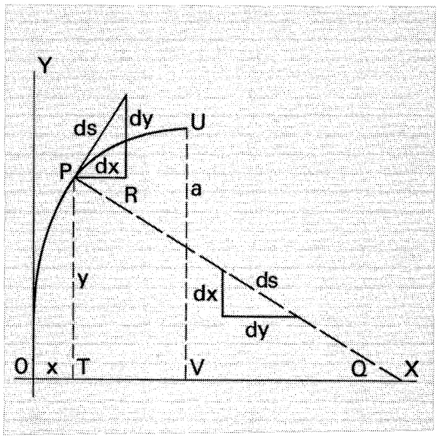
To solve this problem, we first write down the 40 or so factorizations of 1296, and then consider the information that each of the four answers gives us:

1. If he says none is older than I am, there are several possible factorizations if the informant is 19: $9 \times 9 \times 16$, $8 \times 9 \times 18$, and $6 \times 12 \times 18$. The older the informant, the more factorizations.
2. If he says one is older than I am, and if the informant is 432 to 647 years, then there is just one factorization: $1 \times 2 \times 648$. Otherwise there are multiple factorizations (down to age 2).
3. If he says two are older than I am, and he is 27 to 35, then the only factorization is $1 \times 36 \times 36$. Older than 35 is impossible, and every age 19 to 26 admits multiple factorizations (e.g., $1 \times 27 \times 48$).

Since we assume the informant is under 432 years, condition 3 is the only one admitting a unique factorization. The house number is then $1 + 36 + 36 = 73$. But neither this problem nor the original explains how the census taker matched the correct age with each of the three names.

Also solved by R. Robinson Rowe, Smith D. Turner, and Patrick J. Sullivan.

16 Find a curve having nonconstant radius of curvature such that all the centers of curvature lie on the x axis.



The only solution is from R. Robinson Rowe, who rephrases the problem to find a curve with its evolute on the x axis:

Let $y = F(x)$, $y' = dy/dx$, $y'' = d^2y/dx^2$, $ds^2 = dx^2 + dy^2$, $s' = ds/dx = \sqrt{1 + y'^2}$. In the figure, at P on the curve the radius of curvature $R = PQ$ is normal to the curve, and the center of curvature Q is required to be on the x axis. From similarity of triangle PQT to the differential triangle,

$$R = y\, ds/dx = ys' = y\sqrt{1 + y'^2}. \quad (1)$$

From calculus, the general equation for curvature is

$$R = (1 + y'^2)^{3/2}/y''. \quad (2)$$

Equating (1) and (2),

$$y\sqrt{1 + y'^2} = (1 + y'^2)^{3/2}/y''$$

$$yy'' = 1 + y'^2 \quad (3)$$

With boundary conditions $x = y = 0$, $y' = \infty$, and the first integration yields

$$y' = \sqrt{a^4 - y^4}/y^2 \quad (4)$$

defining an upper boundary at $y = a$ where $y' = 0$. It will be expedient to deal with a unit curve from which all others may be derived by a scale factor. Making $a = 1$ and separating the variables,

$$dx = y^2 dy/\sqrt{1 - y^4}. \quad (5)$$

This elliptic integral of the second kind may be solved by series or by elliptic functions. For the series solution,

$$dx = y^2(1 - y^4)^{-1/2}dy = (y^2 + \tfrac{1}{2}y^6 + 3y^{10}/8 + \ldots)dy \quad (6)$$

$$x = y^3\,(1/3 + y^4/14 + 3y^8/88 + y^{12}/48 + 35y^{16}/2{,}432 + \ldots). \quad (7)$$

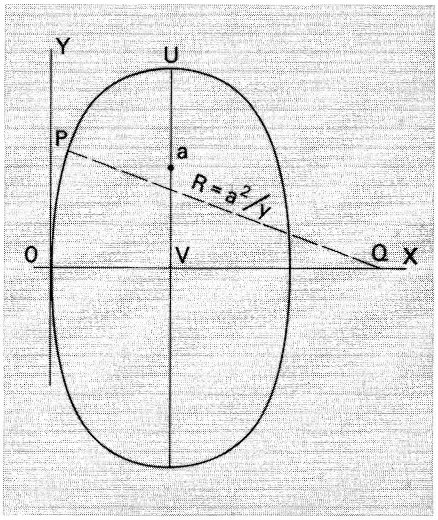
And for the so-called exact solution, the transformation

$$y = (\sin\phi)/\sqrt{1 + \cos^2\phi},\ dy/d\phi = (2\cos\phi)/(1 + \cos^2\phi)^{3/2},\ 1 + \cos^2\phi = 2(1 - k^2\sin^2\phi),\ k^2 = \tfrac{1}{2} \quad (8)$$

converts (5) to a tabular form and direct solution:

$$x = k^3 \int(\sin^2\phi\, d\phi)/(1 - k^2\sin^2\phi)^{3/2} \quad (9)$$

$$= \sqrt{2}\,[E(\phi,k) - \tfrac{1}{2}F(\phi,k)] - y\cos\phi. \quad (10)$$



Equation (10) defines the curve from O to U, the latter point being located by $y = 1$ and the complete elliptic integrals,

$$x_u = \sqrt{2}\,(E - \tfrac{1}{2}K) = 0.599\,070\,1202. \quad (11)$$

Obviously, by reflections, the arc OU is only one quadrant of the entire curve (drawn below), a flat-sided oval centered at V, with a major diameter of 2 and a minor diameter near 1.2.

After all this, Mr. Robinson is kind enough to say, "I enjoyed this incentive to review elliptic integrals, after 50 years of neglect."

17 I placed 15 dimes and 15 nickels in six cups such that each cup contained the same number of coins but a different amount of money. I made six labels showing correctly how much money each cup contained, but attached to each cup an incorrect label. I explained the situation to six logicians and gave a cup to each. I asked each man in turn to feel the size of as many coins as he wanted in his own cup and announce something interesting. The only evidence each man had was the size of the coins he felt, the incorrect label on his own cup, and the statements made by those who preceded him. The first man said, "I feel four coins which are not all the same size; I know that my fifth coin must be a dime." The second man said, "I feel four coins which are all the same size; I know that my fifth coin must be a nickel." The third man said, "I feel two coins, but I shall tell you nothing about their size; I know what my other three coins must be." The fourth man said, "I feel one coin; I know what my other four must be." The problem is to determine how the remaining two cups were labeled and what the total value of the money in those two cups was.

The following is from Captain John Woolston:

The first man knew what coins he had. The other men knew that he had: (a) three nickels, two dimes, and a 30¢ label; (b) two nickels, three dimes, and a 35¢ label (which he knows himself); or (c) one nickel, four dimes, and a 40¢ label.

The second man also knew what he had, and everyone else knew that he had either (a) four dimes, a nickel, and a 50¢ label (as he knows himself) or (b) five nickels and a 30¢ label. Incidentally, his answer is independent of the first man's answer.

The third man had to think and trust the first two men. What he felt was a nickel and a dime, and his label said 30¢. Consequently, he knew that the second man had 45¢ and a 50¢ label (since he held the 30¢ label himself), so that the first man had 40¢ and a 35¢ label (since the second man held 45¢ and he held the 30¢ label). Knowing that he had a nickel and a dime (which he needed to know to eliminate 25¢ and 50¢), his possibilities were (a) 30¢, which is impossible since he had a 30¢ label; (b) 35¢, which he actually had; (c) 40¢, which he couldn't have since the first man had it; or (d) 45¢, which he couldn't have because the second man had it. The other men know that this combination is the only one the third man can have which would allow him to know his holdings after feeling only two coins. This is because two coins can only eliminate 25¢ and 30¢ or 25¢ and 50¢ or 45¢ and 50¢, and the 30¢ label is the only one which brings the possibilities within the range.

So now everyone knows what the first three men have—both coins and labels. The fourth man felt a nickel and saw a 25¢ label. He knows he cannot have 50¢ because of his nickel and he cannot have 25¢ because of his label, so he must have 30¢. Everyone else knows that he has 30¢ and a 25¢ label, since one coin can only eliminate 25¢ or 50¢, and neither of the other two labels brings a solution within range. At this point, the last two men have 50¢ and 25¢ and the 40¢ and 45¢ labels for a total of 75¢. They also know this without feeling and without looking at their labels; hence, one feel and they know which coins they have.

Also solved by James W. Dotson, Stanley Horowitz, R. Robinson Rowe, Kenneth L. Zwick, and the proposer, David P. Dewan.

18 Fill in the digits in this multiplication problem, using each of the 10 digits (0, 1, 2, . . . 9) exactly twice:

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  xxx
  xxx
 -----
  xxx
 xxx
xxx
-----
xxxxx
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The following is from Hubert du B. ("Lucky") Lewis:

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  179
  224
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  716
 358
358
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40096
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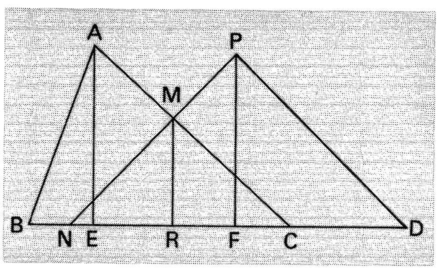
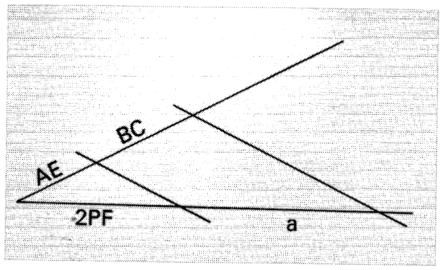
By inspection, the multiplicand must be less than 400. By quick elimination, the multiplicand must be one-hundred-something, and the multiplier must then be 223 or higher. It was not 223, and 224 came next.

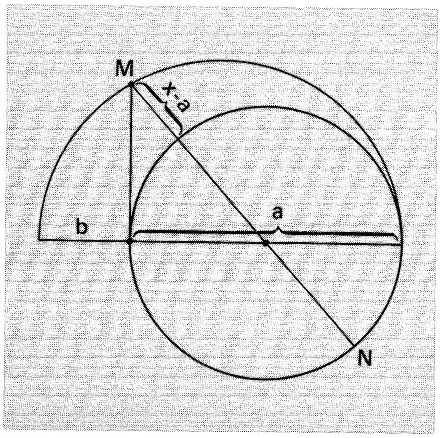
Also solved by Richard A. Bator, Richard P. Bishop, Robert J. Swaney, R. Robinson Rowe, John P. Rudy, and the proposer, Thomas B. Jabine.

19 Given a triangle ABC and a point P, find a method for constructing a line through P bisecting the area of the triangle.



The following from Smith D. Turner: The given triangle in the drawing is ABC, the given point is P. PN is therefore the required line. PD is parallel to AC. AE, MR, and PF are perpendiculars. On the basis of areas,
2 x MR x NC = AE x BC.
But since triangles MNC and PND are similar,
MR/NC = PF/(NC + CD).
Eliminate MR from this pair of equations and the result may be arranged
NC [NC — (AE x BC)/2PF] = (AE x BC)/2PF x CD,
or, say,
X(X — a) = ab
Now a is merely the fourth proportional to three known lines—AE, 2PF, and BC—and so may be constructed. Therefore ab is a known area, and X may be found—a side of a rectangle where the area and the difference of sides is known. With NC = X known, PN is drawn.





Also solved by R. Robinson Rowe and Captain John Woolston.

20 A said to the farmer, "I know you own a rectangular plot in that 20-by-20 section, and I know the area of your plot. Is the length greater than twice the width?" B said to the farmer, "Before you answer let me state that I knew the width, and I now know the length." C said, "I did not know the length, width, or area; but now I know the dimensions." What are they?

The following is from Christopher Brooks: The trick to solving this problem is to read between the lines. A asks, ". . . Is the length greater than twice the width?" which must be taken to indicate that the length of the plot *could* be greater than twice the width (and fit into a 20 x 20 section). Since $L_{max} = 20$, W must be ≤ 10. Therefore $A \leq 200$. B knew the width, and since $L \geq W$, B knew that $A \geq W^2$. Thus, for B to infer the length of the plot from A's question, we must have $A = W^2 = 200$, and $L = W = 10\sqrt{2}$.

Also solved by James W. Dotson, Stanley Horowitz, Benjamin Fulbright, Captain John Woolston, Richard P. Bishop, and Frank Rubin.

Better Late Than Never

Additional answers have been received on this year's problems, as follow.

1 Frank Rubin

3 Dudley F. Churd has found four other ways to make the hand. Also solved by Winslow H. Hartford, Patrick J. Sullivan and Captain John Woolson

4 John G. Maier, Frank Rubin and Captain John Woolson

5 Frank Rubin

6 James L. Larsen

7 Anonymous and James L. Larsen

9 Frank Rubin and James L. Larsen

10 The proposer, Henry S. Lieberman, notes that Thomas Sadler's proof was incomplete. He writes, "While everything that Mr. Sadler has done is correct, he has not completed the solution. Indeed, Mr. Sadler asserts correctly that the two conjugacy classes of G are $\{e\}$ and $G - \{e\}$. Let's take the first question: If G is assumed to be finite . . . Let $n = \#G$. Then $\#(G - \{e\}) = n - 1$. But the order of a conjugacy class must divide the order of the group. Thus, $n - 1 | n$. Now the only instance in which this can happen is if $n - 1 = 1$, i.e., $n = 2$. Thus $\#G = 2$, i.e., G is the (cyclic) group of order 2. As for the second question, let us make use of Mr. Sadler's correct assertion that $\#G = p^r$, where p is prime. Now it is well known that G has a nontrivial center. But the center is a normal subgroup and therefore the union of conjugacy classes. Therefore, in our case G equals its center and is thus abelian. But the number of conjugacy classes of an abelian group is the same as the order of the group! Whence, again we must have $\#G = 2$. Hence in either case G is the (cyclic) group of order 2."

In later correspondence Mr. Lieberman notes that the above is also not complete and repairs it as follows:
"What I failed to notice is that Mr. Sadler's conclusion that $G = p^r$, p prime, is based on the assumption that G is finite. But to handle the case for which G is assumed to have a nontrivial element of finite order, we cannot also assume that G is finite. For this case let me provide my oft-promised combinatorial proof. I claim that in this case also G is the group of order 2. The proof is as follows: Mr. Sadler has already correctly shown that every nontrivial element of G has the same order m and m must be prime. Assume m is an odd prime. Let $x \neq e$. Then $\exists\, g \neq e \in G$ such that $x^{m-1} = g x g^{-1}$. Hence $(x^{-1})^{-1} = (g x g^{-1})^{-1} = (g^{-1})^{-1} x^{-1} g^{-1}$, i.e., $x = g x^{-1} g^{-1} = [g(g x g^{-1})g^{-1}] = g^2 x g^{-2}$. Indeed, it follows that $x = g^k x g^{-k}$ for all even k. In particular, $x = g^{m-1} x g^{-(m-1)}$, i.e., $x = g^{-1} x g$ or $x = g x g^{-1}$. We thus have $x = x^{-1}$, or $x^2 = e$. But this contradicts that m is odd. Hence, m is an even prime, i.e., $m = 2$. But this x requires that G is abelian, whence we must have $\#G = 2$. This settles the matter."

Allan J. Gottlieb studied mathematics at M.I.T. with the Class of 1967 and is now a Teaching Assistant at Brandeis University. Send answers, problems, and comments to him at the Department of Mathematics, Brandeis University, Waltham, Mass. 02154.