Integral Solutions, Ladders, and Pentagons

Apparently this column is actually read by someone outside my immediate family. It was a pleasant surprise to see "Puzzle Corner" referred to in no less than Martin Gardner's "Mathematical Puzzles and Diversions" appearing in *Scientific American* for March. It is an honor to be mentioned in the most famous puzzle column written in America.

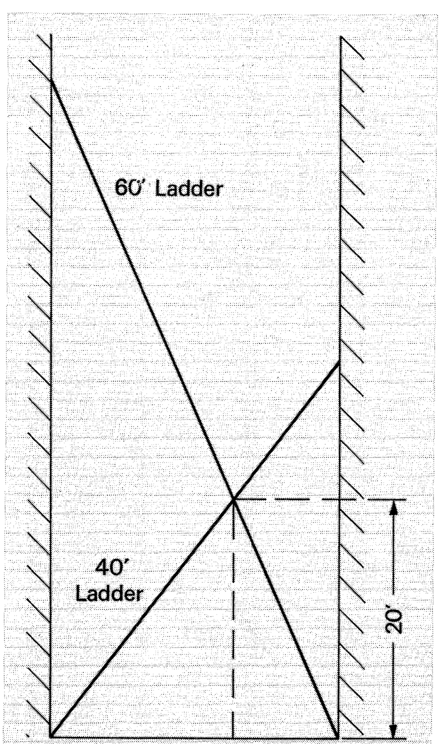
Does this entitle me to a free subscription to *Scientific American*, or must I continue to grub off my wealthy friends?

Problems

26 Frank Rubin wants you to show that there are infinitely many integral solutions to $x^3 + y^3 + z^3 + w^3 = 0$.

The following is from Norman L. Apollonio:

27 A 60-ft. ladder and a 40-ft. ladder intersect 20 ft. above street level. How wide is the street?



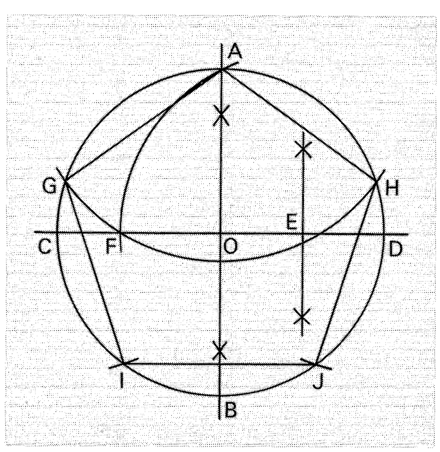
Here are two related problems in one from C. E. Hedrick; he calls them "two concealment ciphers which might be a challenge":

28a "Pediatric researchers find that apparent learned errors attenuate reliable actions, channelling unavoidable patterns at neural circuitry. Assuming aggressive or regressive patterns at an early age tends to reinforce the unreliability."

28b (somewhat harder) "In legal disputes, the most rational attempts to respond to accusations necessitate thorough research, high standards, astute observation, the psychologist's feeling for allusion, and all possible care to avoid countersuits."

29 William J. Wagner wants you to show that given three unequal circles whose centers are noncolinear, the points of intersection (A, B, and C) of the three pairs of common tangents are colinear.

The last problem is from Norman D. Megill, an M.I.T. sophomore:



30 The drawing shows the construction of a pentagon, using only a straight edge and a compass. The construction is as follows:
1. Draw a circle with center at 0.
2. Draw line CD through the center of the circle.
3. Construct the perpendicular bisector to CD, line AB.
4. Construct the perpendicular bisector to OD, dividing it into two equal parts, OE and ED.
5. Place the compass point on E and the lead on A, and draw arc AF.
6. Place the compass point on A and the lead on F, and draw arc GH.
7. Leaving the compass with this setting, place its point on G and locate I on the circle.
8. Leaving the compass with this setting, place its point on H and locate J on the circle.
9. Draw the pentagon using points I, G, A, H, and J.

The problem is to prove or disprove that the figure constructed is a regular pentagon.
If one analyzes the problem trigonometrically, it reduces to showing that

$$\sin 36^\circ = (\sqrt{10 - 2\sqrt{5}})/4. \qquad (1)$$

This I have not been able to prove or disprove. However, using 10-place tables, I find that
$16 \sin^2 36^\circ = 5.527864045\ldots$ and
$10 - 2\sqrt{5} = 5.527864045\ldots$
Can any reader prove or disprove the relationship (1)?

Speed Department

The only two speed problems are very similar and come from Frank Rubin. Critical shortage: send speed problems!

SD10 Arrange four toothpicks to make four isosceles right triangles.

SD11 Now try to get seven using five toothpicks.

Solutions

11 A living group calculated its collective grade point average for the all-campus competition. The average grade point was determined to be 3 1/3. The members of the group were also competing for the *Voo Doo* Highest Average Reciprocal Grade Point Award. This grade point is formed by averaging the reciprocals of the individual grade points.

This living group was the last to enter the competition, and the highest average reciprocal grade point prior to their entry was 0.29. The group had to rush to enter before the competition deadline and had time only to calculate their average grade point of 3 1/3. However, the judge looked at this number and declared the group to be the winner of the *Voo Doo* Award. How did he know?

The following is by James P. Friend: The average grade point =

$$G = 1/n \sum_{i=1}^{n} g_i$$

where g_i are the individual grade point scores and n is the number of individuals in the group. The average reciprocal grade point R is

$$R = 1/n \sum_{i=1}^{n} 1/g_i. \text{ Now}$$

$$RG = 1/n^2 \left[\sum_{i=1}^{n} \sum_{j=1}^{n} g_i/g_j \right]$$

$$= 1/2n^2 \left[\sum_{p=1}^{n} \sum_{q=1}^{n} (g_p/g_q - g_q/g_p) \right].$$

Examine one term in the second double sum:

$g_p/g_q + g_q/g_p = (g_p^2 + g_q^2)/g_pg_q$
$= [(g_p - g_q)^2 + 2g_pg_q]/g_pg_q$
$= [(g_p/g_q) - 1]/(g_p/g_q) + 2.$
Therefore $g_p/g_q + g_q/g_p > 2$.
Since there are n^2 such quantities in the double sum,
$RG > 1$ ($RG = 1$ for $g_1 = g_2 = g_3 = \ldots = g_n$).
The judge knew this relationship; for $G = 3\ 1/3$, $R > 3/10$.

The following alternate solution deserves special merit for something, but its quality will be appreciated only by those familiar with *Voo Doo*, M.I.T.'s late lamented "humor" magazine; it comes from J. Shelton Reed, who was *Voo Doo*'s editor in 1963-64: "The judge knew the living group had won the *Voo Doo* reciprocal grade point award because he was a *Voo Doo* staffer and thus had powers beyond the ken of ordinary mortals . . ."

Also solved by Peter Ross, John Pierce, Mark Yu (who calls himself "the homologized kid"), Gary N. Sherman, Arthur W. Anderson, Captain John Woolston, R. Robinson Rowe, James W. Dotson, and Frank Rubin.

12 On the following bridge hands

♠ A K 9 4 2
♥ A 3
♦ A 6 5
♣ A 7 3

♠ 6 3 ♥ Q J 10 9 8 7 6 5 4 ♦ K ♣ 2 (West)

♠ 10 ♥ — ♦ Q J 10 9 8 4 ♣ Q J 10 9 6 5 (East)

♠ Q J 8 7 5
♥ K 2
♦ 7 3 2
♣ K 8 4

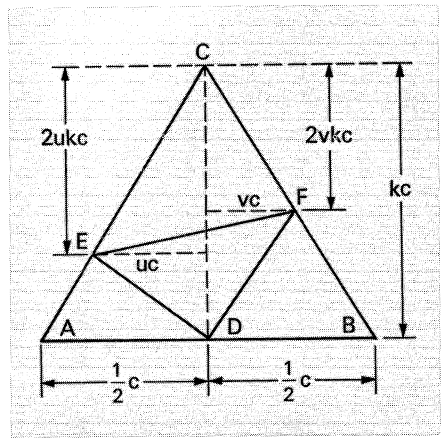
the bidding was West, 4♥, North, double; East, pass; and South, 4♠. Lead is Q♥. Dummy's ace is put up and ruffed by East. Make any return.

The following is from Leon M. Kaatz: The problem is to make 4♠ after the ♥Q is led, covered by the ace in the dummy and ruffed by East with the ♠10. On trick 1, declarer must throw away his ♥K. Whichever minor suit East returns, suppose clubs, is won by dummy's ace at trick 2.
Tricks 3 and 4: ♠Q and ♠J pull trump.
Trick 5: Cash ♦A (or ♣A if trick 2 was a diamond).
At this point the exact location of all the outstanding cards is known. Trick 1 revealed West to have started with nine hearts, and since he has come up with two spades, one diamond, and one club he must have exactly the ♥4 through ♥J left. This leaves East with eight minor suit cards at this point.
Trick 6: Lead out ♥3, forcing West to win.
Trick 7: On the forced heart return a diamond is thrown from North while South ruffs.
Tricks 8 to 10: North cashes three high trumps.
While the last trump is led out (trick 10), North has one diamond, two clubs, and the trump; East has four cards in the minor suits; South has two diamonds and two clubs headed by ♣K (♣A has been cashed); West has four hearts. If, on the lead of the ♠9, East goes down to one club, then South throws off a diamond. Tricks 11 and 12 are now won by South's two high clubs to fulfill the contract. If East goes down to one diamond on the lead of ♠9, then South throws off his low club. A diamond is now led from the North hand, forcing East to win with his singleton. East must now lead a club which South will win with the ♣K. South now cashes the 13th diamond to fulfill his contract.

Also solved by a long list of bridge players: Smith D. Turner, Henry C. Howarth, Captain John Woolston, James W. Dotson, R. Robinson Rowe, Don Williams, Stan Horowitz, Eric Weitz, Paul M. Horvitz, Don Forman, F. W. Milliken, Edward S. Gershung, Bill McClary, E. C. Ingraham, William P. Bengen, C. C. Crystal, Robert H. Park, Edwin Burmeister, P. J. Sullivan, and William G. Kussmaul, Jr.

13 Find conditions on the ratio of the altitude to the base of isosceles triangle ABC such that the inscribed triangle DEF with maximum area (D is at the midpoint of AB) has FE parallel to AB.

The following is from R. Robinson Rowe: The figure published did not agree with the text, as it showed equal angles at E and F, which is not given (though it will follow).



My figure purposely shows these angles unequal and EF not parallel to AB. We are given that triangle ABC is isosceles, that D is at the midpoint of AB, that triangle DEF is so inscribed in ABC as to maximize its area. We are asked what ratio of altitude CD to base AB will make FE parallel to AB.
Let this ratio be k and express dimensions in terms of the base AB = c. Identify E and F by their distances uc and vc from CD. Let the area of DEF = A and ABC = A′. Then $A = A'(u + v - 4uv)$ = a maximum
$dA/du = A'(1 - 4v) = 0$ and $v = 1/4$
$dA/dv = A'(1 - 4u) = 0$ and $u = 1/4$
Therefore $2ukc = 2vkc = \frac{1}{2}kc$ and EF is parallel to AB for any value of k. Also, $A = \frac{1}{4}A'$ for any value of k.

Also solved by Mark Yu, Frank Rubin, Captain John Woolston, and Jerry L. Robertson.

Better Late Than Never

There are further replies on two problems published last year. On problem **29,** Michael Beeler has some different results from those published in December, 1969 as shown below.

Frank Rubin points out that he did *not* solve problem **31,** though we gave him credit for doing so in October/November.

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n	1	2	3	4	5	6	7	8
2	32	34	23	1				
3	215	306	287	83	9			
4	2512	2918	2280	1068	210	12		
5	31062	32664	15764	7935	2190	365	20	
6	368718	339305	105506	57685	22245	4111	2430	
7	4217249	3283295	783760	436801	199956	36708	37002	5229