Bridge, Group Theory, and a Jigsaw Puzzle

In this issue we'll make another effort to catch up on all "better late than never" responses. I'm sorry for the delay, but space considerations which were beyond my control dictated the outcome.

This summer I decided to increase my frustration with life—I started taking golf seriously. After reading Benjamin J. Averbach's article in the May issue of *Technology Review,* which strongly favored aluminum-shafted clubs, I threw the whole bank roll into a complete set of brand new Wilson Staffs. They look and feel beautiful. Two years ago people in my foursome would say, "No wonder you play so badly, your clubs are awful!" Now, after I explain the significance of these aluminum shafts—and that most pros use them—these same golfers remark, "How can you play so badly with such a beautiful set of clubs?" It seems to be just like the amp. of last year. Good equipment in bad hands . . .

There's a critical shortage of Speed Problems. Please help!

Problems

This month's first problem is from Edward Friedman:

6 Given the equation
$(x + 4)(6 - x) = 9$, solve for x.
One method is simply $x + 4 = 9$, $x = 5$; and $6 - x = 9$, $x = -3$; both answers are correct! The problem is to find a general form of the equation for this method of solution to work.

7 The second challenge is a bridge problem from Frank Model:

North:
♠ x x
♥ K Q x x x
♦ x x
♣ A K Q J

West:
♠ A Q x x
♥ x x
♦ Q J 9 x
♣ x x x

East:
♠ J x x
♥ x x
♦ x x x
♣ x x x x x

South:
♠ K x x x
♥ A J x x
♦ A K 10 x
♣ x

The contract is 6 no trump by South, and the opening lead is ♦Q.

Here is the first "jigsaw puzzle" ever in Puzzle Corner. My thanks to Smith D. Turner:

8 If it takes an hour to work a jigsaw puzzle of 100 pieces, how long should it take to do one with 300? (To simplify, put all solvers on the same basis, assume that the puzzles have no regular edges and are of solid color, and that pieces are of similar size and pattern in both puzzles.)

9 Brad Yundt wants you to connect the three midpoints of a triangle and prove that each of the resulting line segments is parallel to and equal in length to one-half of the opposite side.

Henry S. Lieberman submits the following problem in elementary group theory:

10 Let G be a group with precisely two conjugacy classes (x and y are conjugate if there is an element a such that $y = a x a^{-1}$). If G is assumed to be finite, what can be concluded about G? If instead we assume that G contains a nontrivial element which is of finite order (i.e., there is an x not equal to 1 in G and that $x^n = 1$ for some nonzero integer n), what can be concluded about G?

Speed Problems

SD3 Prove using all available statistics that the Mets cannot possibly beat the Orioles. Send all solutions to Earl Weaver, Baltimore, Maryland. P.S. We're number one!

The following is from John M. Taylor; he calls it "another punctuation problem:"

SD4 John where James had had had had had had had had had had had the teacher's approval.

Better Late Than Never

9 Richard A. Bator and Lars H. Ljodahl.

10 Robert G. Hall and James J. Heyman.

11 Find the range of positive values of x such that, given x, the only positive value of y which satisfies $y^x = x^y$ is the trivial solution $y = x$. For those values of x for which nontrivial solutions for y exist (e.g., $x = 2$, $y = 4$), how many solutions are there for a given x? If a value of x is selected at random from the open interval (o,e), what is the probability that a nontrivial solution for y lies in the same interval?

Peter Ross has given explicitly the unique nontrivial solution of $x^y = y^x$ with $1 < x < e$. He notes that for $1 < r < \infty$, the following yield *all* nontrivial positive solutions:

$$x = r^{\frac{1}{r-1}}$$

$$y = r^{\frac{r}{r-2}}$$

There are negative solutions like $(-2)^{-4} = (-4)^{-2}$.

12 F. J. Heath has given a computerized solution.

13 George Martin has responded.

14 Let $n_1 = x^x$, $n_2 = n_1^x$. . . .

$$n_i = (n_{i-1})^x = x^{x^{\cdot^{\cdot^{x}}}}$$

and let $N = \lim n_i$ as i grows large beyond bound. Do real values of x exist for all positive N? If so, what is the relationship? If not, what are the limitations?"

Richard Lopes submits the following: "The published solution (*see* Technology Review *for April, page 99)* is incorrect, since if
$N = \lim_{i \to \infty} n_i$, then $x = N^{1/N}$ which has real values for all positive N. The statement, 'Surely if $x > 1$, $N = \infty$' is absurb.
As a counter-example, notice when
$N = 2 \neq \infty$, $x = \sqrt{2} = 1.414 \ldots > 1$."

The following is from Frank Rubin:

Unfortunately, Mr. Yu's solution is largely nonsense. For the case $x > 1$, Mr. Yu states that N is always ∞. Nonsense!

Consider $x = \sqrt{2}$. Then
$n_1 = x^{\sqrt{2}} < x^2 = 2$
$n_2 = x^{n_1} < x^2 = 2$
$n_3 = \ldots$
Thus $\{n_i\}$ is monotone increasing and bounded above by 2. So $\{n_i\}$ converges! In fact, $N = \lim n_i = 2$. The fact is that the limit exists whenever $0 < x < e^{1/e}$, as my solution showed, and I believe it also converges for $x = e^{1/e}$ to N:e.

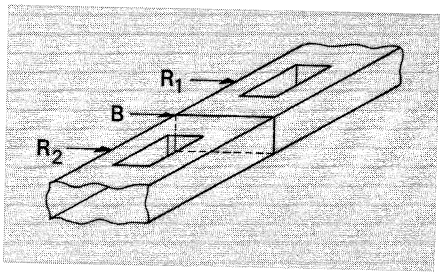
For the case $0 < x < 1$, Mr. Yu states that $\{n_i\}$ is monotone increasing, and he bases his proof on this assumption. Again nonsense! For $0 < x < 1$ the sequence $\{n_i\}$ is oscillating. The correct solution is that x exists when $0 < N \leq e$.

John W. Langhaar has also responded

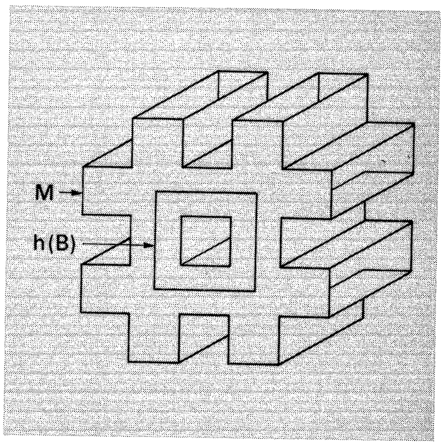
16 James Bradley and Dennis Ribler at Rochester noticed that the double infinite torus and the single infinite torus are different, since any compact subset of a single infinite torus is contained in a compact subset with connected complement, and the double infinite torus does not have this property. But they could not show that the single infinite torus and the jail cell are the same. A final hint: use classification thin for *compact* two manifolds with boundary.

The correct answer was found by Edward C. Hendricks:

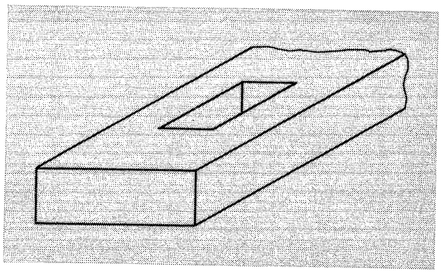
It appears to me that the intersecting bars surface and the double infinite torus are not homeomorphic, contrary to the statement in the May issue. For suppose, calling the surfaces S and T, respectively, that $h : T \rightarrow S$ is a homeomorphism.

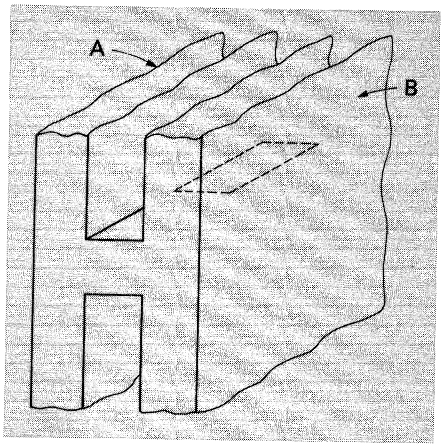


Let R_1 and R_2 be the submanifolds shown in the drawing and let B be their common boundary (R_1 and R_2 are relatively open and are disjoint from their common boundary B), so that T is the disjoint union of R_1, R_2, and B. Then h(B) is compact, since B is. Thus we can find a compact subset M of S such that M contains h(B) and S—M is connected. Since S—M is connected it is contained in one of the two disjoint connected components, $h(R_1)$ and $h(R_2)$ of S—h(B).

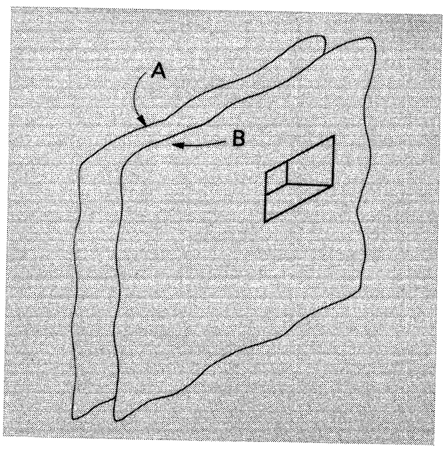


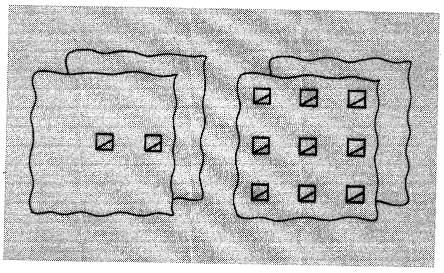
Suppose $S—M \leq h(R_1)$. Then $h(R_2) \leq S—h(R_1) \leq M$. Thus $h(BuR_2) \leq M$. But $h(BuR_2)$ is closed since BuR_2 is closed, and thus $h(BuR_2)$ is compact. But $h(BuR_2)$ is not compact, since BuR_2 is not compact. Similarly if $S—M \leq h(R_2)$. It appears from the sequence of manifolds in the following drawings that the single infinite torus is homeomorphic to S.





The following can be obtained by folding out the sheets A and B of the previous drawing.





The left figure, above, is the previous drawing on a smaller scale.

Robert C. Bell also comments on this problem, noting that my previous hints were incorrect:

Let a "cutting circle" mean a homeomorph of the circle which is not contractable to a point on the surface in question. Then no cutting circle divides the infinite jail cell surface into two pieces. There are infinitely many such cutting circles which divide each of the other two, but in every case such a division of the infinite holed torus which only extends to the right leaves one component with a finite Betti number, while in each case such a cut in the infinite holed torus extending both to the left and to the right leaves both pieces with an infinite Betti number. Therefore no two are homeomorphic.

18, 19 Frank Model has responded.

21 Robert G. Hall has responded

26 Robert L. Bishop, James L. Heyman, and Captain Jerry L. Robertson.

27 Smith D. Turner.

28, 29 Finally someone—Samuel S. Wagstaff, Jr.—has solved an Egendorf problem:

Define f on non-negative integers n by f(n), set equal to the product of the digits of n (when n is written to the base 10). It is plain that $f(n) < n$ when $n \geq 10$ and that $f(n) = n$ when $0 \leq n < 10$. Thus it makes sense to define the *length of a positive integer n* to be the number of iterations of f required to get to a one-digit number. Now $f(77) = 49$, $f(49) = 36$, $f(36) = 18$, and $f(18) = 8$, so 77 has length 4. In problem 28 we must show that every other two-digit number has length less than 4. Clearly the numbers 10 through 24 have length 1. The numbers n of length 2 are just those n such that f(n) has length 1, so we ask which n have $f(n) \leq 24$, and we see easily that all $n \leq 46$ except $n = 39$ have $f(n) \leq 24$, i.e. length ≤ 2. Also note that the numbers

48, 54, 56, 63, 64, and 72 (1)

have length 2. The only numbers n such that $f(n) > 46$ are the numbers 68, 69, 77 through 79, 86 through 89, and 96 through 99. But, with the exception of 77, f of these numbers is in (1), so the length is ≤ 3.

As for problem 29, I found no general method but used a computer to go up to 10,000.

Here is a table showing how many n-digit numbers have length m:

n/m	0	1	2	3	4	5	6
1	10*	0	0	0	0	0	0
2	0	32	34	23	1	0	0
3	0	223	304	281	83	9	0
4	0	2524	3052	2134	1068	210	12

*counting 0 as a one-digit number.

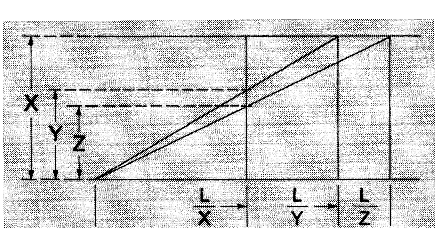
The nine 3-digit numbers of length 5 are: 679, 688, and rearrangements thereof (such as 976, 868, etc.), and the 12 4-digit numbers of length 6 are 6788 together with its rearrangements.

The only other comment I have is that the image of f is the set of all positive integers whose only prime factors are 2, 3, 5, and 7, and that this is a very thin set of integers. Large integers are very rarely in this set, and the sum of the reciprocals of all the numbers in the set is finite (namely, 35/8). This leads me to suspect that perhaps there is an upper bound on the length of an integer, but I hesitate to make a formal conjecture in view of the evidence in the above table. As Snoopy once said of Charlie Brown in a similar situation, "How wishy-washy can you get?" Anyway, if any of your readers can find numbers with length greater than 6 or a way of constructing numbers of arbitrarily large length, I'd like to know it.

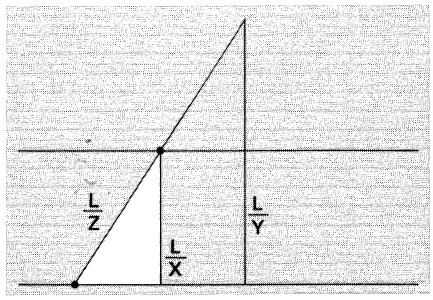
30 Robert S. Cox.

33 Construct a triangle given the three altitudes.

This one provided little trouble for Captain John Woolston; who writes: "As soon as I stopped fiddling with my pencil and engaged my mind, the answer became simple. Since obviously the area of the triangle is constant regardless of which base and altitude you use, the sides of the triangle are inversely proportional to the altitudes and adjusting to one altitude gives the final triangle. Arriving at sides bearing an inverse relationship to altitudes, there are several approaches; but I prefer to construct as follows: Superimpose the three altitudes with bases together with a pair of parallel lines perpendicular to the altitudes through the two ends of the largest altitude, then pick any old point on the base line and draw inclined lines through the ends of the altitudes across the other parallel line and drop perpendiculars to the base as follows:

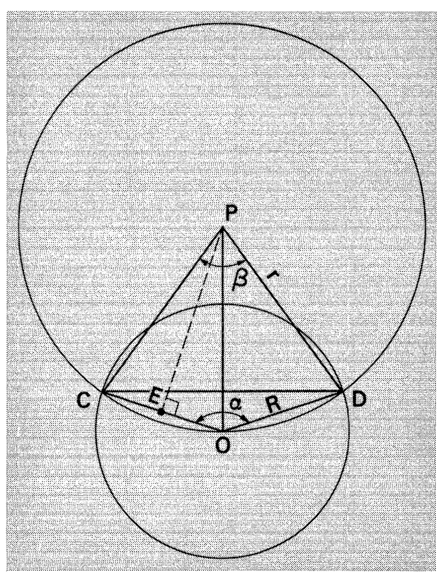


"The three lengths on the base line are used to construct a triangle, and it is made larger or smaller by moving one side parallel to itself to suit one altitude; and this is the desired triangle as follows:



"Of course, it might be wise to point out to the neophyte geometry teacher that he must check the lengths of his altitudes before he assigns the problem to his students. While it makes no difference how big Papa Bear is (the largest altitude) nor how small Baby Bear is (the smallest altitude) if Mama Bear (the middle sized altitude) is too big, you won't have a family (the triangle doesn't close). Mathematically, if X, Y, and Z are the lengths of the altitudes in descending order of lengths, then $1/X + 1/Y > ½$, or $Y < ZX/(X - Z)$. Failure to note this could cause a red face, though I hope not student riots."

Also solved by Sunney D. Alexis, James J. Heyman, Mrs. Mary Lindenberg, R. Robinson Rowe, Smith D. Turner, Samuel S. Wagstaff, Jr., and Mark Yu.



35 A cow is grazing in a circular field A of any given area, say 10 acres. She is tied at O with a chain R long. How long must the chain be for her to cover an area B of one acre?

A very clear answer came from Donald E. Savage:
As the chord CD makes evident, the grazing area is the sum of two circular segments. Therefore the grazing area is $R^2/2\,(\alpha - \sin\alpha) + r^2/2\,(\beta - \sin\beta)$.
(The formula for the area of a circular segment may be cribbed from Burington's Tables or separately derived.) But from the triangle OEP we see that
$\beta/4 + \alpha/2 = \pi/2$, and
$\sin(\beta/4) = (R/2)/r$.
Therefore the (grazing area)/(field area) is
$(2/\pi)(R/2r)^2\,(\alpha - \sin\alpha)$
$+ (1/2\pi)(\beta - \sin\beta)$
$= (1/\pi)[2\cos^2(\alpha/2)]\,(\alpha - \sin\alpha)$
$+ \pi - \alpha + (\sin 2\alpha)/2$
$= 1 + (1/\pi)\,[(1 + \cos\alpha)\,(\alpha - \sin\alpha)$
$- \alpha + \sin\alpha\cos\alpha$
$= 1 - (1/\pi)\,(\sin\alpha - \alpha\cos\alpha)$.
For this problem, grazing area/field area $= 1/10$, or
$(\sin\alpha - \alpha\cos\alpha) = .9\pi$.
This can be solved only by numerical methods. Using my slide rule, I find after five tries,
$\alpha = 2.668$ (= 152.8°); therefore $\beta/4 = 90° - 152.8°/2 = 13.6°$.
Therefore $R/r = 2\sin 13.6° = .4704$.
But $\pi r^2 = (10\text{ acres}) = 435600\text{ ft.}^2$, or
$r = \sqrt{138650} = 372.2$ ft. Therefore the length of the chain, $R = (372.2)(.4704) = 175$ ft.

Also solved by Lionel S. Goldring, William R. Osgood, R. Robinson Rowe, Smith D. Turner, and Samuel S. Wagstaff, Jr.

30 Consider the function N(R,x) defined (for R a positive integer and x a complex) by
$N(1,x) = x^x$
$N(2,x) = x^{N(2,x)}$
. . .
$N(k,x) = x^{N(k,x)}$
Can it be continuously extended to R to be any real (or perhaps any complex) number?

This was solved only by the proposer, Douglas J. Hoylman:
We have $\dfrac{1}{e^x - 1} = \sum_{n=1}^{\infty} e^{-nx}$ (proof: sum the geometric series), and

$$\int_0^\infty x^3 e^{-nx}dx = \frac{x^3 e^{-nx}}{n}\Bigg|_0^\infty + \int_0^\infty \frac{3x^2 e^{-nx}\,dx}{n} = 0 + \frac{3x^2 e^{-nx}}{n^2}\Bigg|_0^\infty + \int_0^\infty \frac{6xe^{-nx}\,dx}{n^2} = 0 + \frac{6xe^{-nx}}{n^3}\Bigg|_0^\infty + \int_0^\infty \frac{6e^{-nx}\,dx}{n^3} = 0 + \frac{6}{n^4}$$

(integrating by parts)

$$\text{so } \int_0^\infty \frac{x^3}{e^x - 1}\,dx = \sum_{n=1}^{\infty} \int_0^\infty x^3 e^{-nx}\,dx = \sum_{n=1}^{\infty} \frac{6}{n^4}$$

and, as every schoolboy knows, $\Sigma 1/n^4 = \pi^4/90$, so the solution is $\pi^4/15$.

A late reply from Mark Yu.

37 Evaluate

$$\int_0^\infty \frac{x^3dx}{e^x - 1}.$$

The following solution is from Arthur Gelb: Multiply the numerator and denominator by e^{-x}. This yields

$$I = \int_0^\infty (x^3 e^{-x})/(1 - e^{-x})\,dx.$$

The integrand is well behaved as $x \to 0$; for all other values of x we see that $e^{-x} < 1$. Thus we expand the denominator as follows:

$$I = \int_0^\infty x^3 e^{-x} (1 + e^{-x} + e^{-2x} + \ldots)\,dx$$

$$= \int_0^\infty x^3 e^{-x} \left[\sum_{n=10}^{\infty} e^{-nx} \right] dx$$

$$= \sum_{n=1}^{\infty} \int_0^\infty x^3 e^{-nx}\,dx.$$

The integral under the summation is repeatedly integrated by parts ($u = x^3$, $dv = e^{-nx}\,dx$, etc.), yielding

$$I = \sum_{n=1}^{\infty} 3/n \int_0^\infty x^2 e^{-nx}\,dx$$

$$= \sum_{n=1}^{\infty} 3/n \cdot 2/n \int_0^\infty x e^{-nx}\,dx$$

$$= \sum_{n=1}^{\infty} 3/n \cdot 2/n \cdot 1/n^2 = 6 \sum_{n=1}^{\infty} 1/n^4.$$

The series has a known summation, namely

$$\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90.$$ Thus, for the result we get $I = \pi^4/15 = 6.493 \cdots$

Also solved by R. Robinson Rowe, Donald Savage, W. Allen Smith, Samuel S. Wagstaff, Jr., James A. Waletyko and Mark Yu.

38 Show that

$$\frac{abc,abc}{77}$$

is always an integer.

R. Robinsón Rowe had so little trouble that he felt this should have been a speed problem. He notes that $abc,abc/77 = (1001)abc/77 = 13abc$, an integer.

Also solved by Dermott A. Breault, Michael L. Burach, Lance Draper, Edward Friedman, John H. Goncz (who wrote from Australia), James J. Heyman, Deena A. Koniver, Mrs. Mary Lindenberg, Philip M. Roth, Frank Rubin, John Rudy, Leslie Servi (ninth grade), W. Allen Smith, Daniel H. Sparrel, Neil Steinmetz, Samuel S. Wagstaff, Jr., Jared Wolf, and Mark Yu.

39 Prove that

$$\sum_{i=1}^{n} i^3 = \left[\sum_{i=1}^{n} i \right]^2.$$

or give a counterexample.

Frank Rubin has supplied us with the following
proof by induction:

The statement is clear for $n = 1$. Let L(n) and R(n) be the left-hand and right-hand sums, and

$$S(n) = \sum_{i=1}^{n} i.$$ Then

$L(n + 1) - L(n) = (n + 1)^3$.
We will show that R(n) obeys the same recursion:
$R(n + 1) - R(n) = [S(n) + n + 1]^2 - S(n)^2$
$= 2(n + 1)\,S(n) + (n + 1)^2$
$= 2(n + 1)\,[n(n + 1)]/2 + (n + 1)^2$
$= (n + 1)^2\,(2n/2 + 1) = (n + 1)^3$.

Also solved by Robert A. Bender, Dermott A. Breautt, Michael L. Burack, Jon A. Davis, Lance Drager, Donald Fausett, Edward Friedman, Arthur Gelb, Lt. Randall V. Gressang. James J. Heyman, Anthony W. Merz, Rober Milkman, J. Pelissiei and P. Bergh (jointly), Philip M. Roth, R. Robinson Rowe, Frank Rubin, John Rudy, Donald E. Savage, Homer D. Schaaf, Michael Schaeffer, W. Allen Smith, Daniel H. Sparrel, Neil Steinmetz, Samuel S. Wagstaff, Jr., John Yoachim and Robert Gottlieb (jointly), and Mark Yu.

40 Plot, using Cartesian coordinates, the following equation:

$$\left[2y - \frac{5|x|}{1 + e^{A(|x|-16)}} - \frac{\sqrt{1 + \frac{10}{A} - \left(\frac{x}{16}\right)^2}}{A} \right] \left[A^2(x - 30)^2 + \left(\frac{y}{20} - 1\right)^2 - 1 \right] \left[y - 40 + 40\,e^{-A(x-53)^2} + \frac{\sqrt{1 - \left(\frac{X - 53}{16}\right)^2}}{A} \right] = 0$$

where A is some large positive number, such as 1000.

The three answers received were all different. The proposer's (Donald E. Savage) is given for reasons that will be obvious:

As is usual, the words "Plot . . . the following equation" refer to plotting all *real* number pairs (x,y) that satisfy the equation. Since the right hand side is zero, any number pairs that make a bracket zero will satisfy the equation, so that one can plot each bracket separately. Setting the first equal to zero and rearranging,

$$y = \frac{5/2\,|x|}{1 + e^{A(|x|-16)}} + \frac{\sqrt{1 + 10/A - (x/16)^2}}{2A}$$

Since A is so large, it is evident that the first term is approximately $5/2\,|x|$ for all $|x|$ more-than-a-hair less than 16, and is nearly zero for all $|x|$ more-than-a-hair more than 16. In the vicinity of 16, the first term travels between these two extremes. The second term is a trick to limit the range of x. Since the first term is real for all real x, the second cannot become complex without making y complex. Thus the quantity under the radical sign must stay positive, i.e., $|x| \leq x_1$, where $x_1 \doteq 16 + 80/A$. The second term is very small throughout this range. At x_1, the denominator of the first term is approximately $1 + e^{80}$, which is enormous. Summing up, the first bracket may be sketched as

M

The second bracket, set to zero, is the equation of skinny ellipse. Remembering that the equation of an ellipse with axes along the coordinate axes is
$(x/a)^2 + (y/b)^2 = 1$,
we see that this ellipse is centered on the point ($x = 30$, $y = 20$) and has half axial lengths of 1/A and 20. Thus this bracket's plot may be sketched as

I

The third bracket is a skinny inverted Gaussian function, with origin at the point ($x = 53$, $y = 40$), and height = 40. (The radical term is just my trick to limit the range of x to 53 ± 16.) Thus this bracket may be sketched as

T

Putting them all together, we get

M I T

(rah! rah!).

SD9 Although I normally do not acknowledge responses to speed problems, I will make this exception; Sue Kayton has solved SD9 "without any help." (She is 11 years old!)

Allan J. Gottlieb, who studied mathematics at M.I.T. with the Class of 1967, is a Teaching Assistant at Brandeis University. Send answers and problems to him at the Department of Mathematics, Brandeis University, Waltham, Mass., 02154.