Puzzle Corner

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Year-end report on amplifier: My new roommate, Ron Kadomiya (M.I.T. '67) is a mechanical engineer at Raytheon here in Waltham. He came over and we traced the trouble to the right channel (the same channel on which Dynaco previously replaced a capacitor). He gave the amplifier to an electrical engineer friend of his at the laboratory who ascertained that four—count 'em, four —transistors were blown. Of course the two output transistors were among them. I have purchased replacements and he is to check everything out before I reuse the amplifier. We're so hopeful now that Ron is thinking about buying speakers. I find this definitely a better idea than my buying the speakers.

As promised, no new problems this month, since the answers could not be published until fall.

Solutions

25 Let N be the number of positive integers which contain no digit more than once when they are expressed in base b where b is an integer greater than 2. Show that N is always composite.

The following is from Stephen Owades, who is a freshman at M.I.T.:
"In a base b, one can have positive integers of from 1 to b digits in length without repeating digits. There are (b — 1) ways of filling the first digit (1 through b — 1), (b — 1) ways for the second (0 through b — 1 less the one in the first position), (b — 2) for the third, and so on through (b — k + 1) for the last, where k is the number of digits. Summing for all k, we get

$$N = \sum_{k=1}^{b} (b-1)\,[(b-1)(b-2) \ldots (b-k+1)]$$
$$= (b-1) \sum_{k=1}^{b} [(b-1)(b-2) \ldots (b-k+1)]$$

For $b > 2$, $(b-1) > 1$, and it is obvious that the sigma expression is likewise > 1. Therefore, N is the product of two integers greater than 1 and thus is composite."

Also solved by Eric Hovemeyer and Messrs. Friedman, Grant, Hauser, Karger, Prussing, Ross, and Rubin.

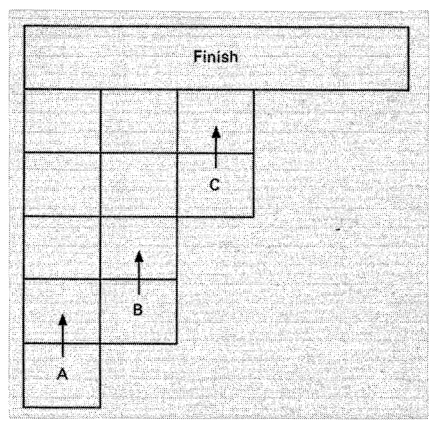
26 Find the smallest integers m and n such that $m - n^3$, m, and $m + n^3$ are all perfect squares.

Douglas J. Hoylman solved this one. He writes:
Let
$m - n^3 = p^2$, $m = q^2$, and $m + n^3 = r^2$.
Then we obtain the equations
$q^2 - p^2 = n^3$,
$r^2 - q^2 = n^3$, and
$r^2 - p^2 = 2n^3$.
Since $r^2 - p^2$ is even, r and p must be of the same parity. But since even and odd squares, respectively, are congruent to 0 and 1 mod 4, $r^2 - p^2$ must be divisible by 4. Hence n^3, and thus n, is even. Now suppose n is not divisible by 3. Then neither is n^3 or $2n^3$. If the difference of two squares is not divisible by 3, it is easily seen that one of the two numbers must be divisible by 3 and the other one not. But this cannot be true of all three pairs (p,q), (p,r), and (q,r). So we derive a contradiction, and n must be divisible by 3. Hence n is divisible by 6. We try the smallest possible value, $n = 6$. Then we must find integral solutions of $r^2 - p^2 = 432$. Factoring the left side, we see that we must have
$2r = (r - p) + (r + p)$
$= 2^a 3^b - 2^{(4-a)} 3^{(3-b)}$
where a = 1, 2, or 3 and b = 0 or 1. (We exclude a = 0, 4 because the right side must be even, and b = 2, 3 by symmetry.) Of the six values for r thus obtained, the smallest is r = 21 (from a = 3, b = 1). Then we obtain p = 3, q = 15, and m = 225. This is clearly the smallest solution.

Also solved by Robert G. Hall, Hubert du B. Lewis, Captain George Martin, Robert G. Mays, John E. Pruessing, and T. Terwilliger.

27 Smith D. Turner wrote, "For betting purposes, a horse race is sometimes simulated by having a number of wooden horses run a course of several moves, the one to move each time being determined by lot. For example, use six horses, throw a die, and the first whose number is thrown (say) 10 times wins. To make this more interesting, I have set it up where one horse must move only a few times, and others increasingly more, to win—thus creating a 'favorite' and 'long shots' in the betting—say 2, 3, 5, 6, 8, and 10 moves with six horses.

"I have found it extremely difficult to calculate the probabilities for such a set-up. Even with a much simplified race—e.g., three horses having to move 5, 4, and 2 times (below)—the calculation was very laborious. In the case illustrated, I get the probability of the favorite C to win as 15001/19683.



"Could anybody check this and—more importantly—come up with a method, computer or otherwise, of handling a more complicated race, say the 2, 3, 5, 6, 8, 10 above?"

Here are the best parts of two solutions received. Harold D. Shane supplies the method:
Let us have k possible outcomes to an experiment, and let the jth outcome have probability P_j, $j = 1, \ldots, k$ $\quad P_1 + P_2 + \ldots + P_k = 1$. Let $t_{n,j}$ equal the number of occurrences of outcome j in n trials. Let n_j be the number of outcomes needed for the "jth horse to win." Then
Pr {jth horse wins on trial n}
= Pr {outcome j on trial n} ×
Pr $\{(t_{n-1,j} = n_j - 1)$
$(t_i \leq n_i - 1 \quad i = 1, \ldots, k \; i \neq j)\}$
Let $P_{n_j j}$ be the probability given above. Now

$$n_j \leq n \leq n_j - \sum_{\substack{i=1 \\ i \neq j}}^{k} (n_i - 1)$$

$$n_j \leq n \leq \sum_{i=1}^{k} n_i - (k-1)$$

Let $N = \sum_{i=1}^{k} n_i$, then

$n_j \leq n \leq N + 1 - k$

To somewhat simplify the notation, let us assume that we shall renumber the horses so that we always want $P_{n,k}$, so that

$$\text{Pr \{kth horse wins\}} = \sum_{n=n_k}^{N+1-k} P_{n,k}$$

Now,

$$P_{n,k} = P_k \sum_{r_1=0}^{n_1-1} \cdots \sum_{r_{k-1}=0}^{n_{k-1}-1} \binom{n-1}{r_1, r_2, \ldots, r_{k-1}, n_k - 1} P_1^{r_1} \ldots P_{k-1}^{r_{k-1}} P_k^{n_k-1}$$

where

$$\binom{n-1}{r_1, r_2, \ldots, r_{k-1}, n_k - 1}$$

is to be interpreted as zero whenever $r_1 + r_2 + \ldots 4r_{k-1} \neq n - n_k$.

For the problem described,

$P_1 = P_2 = \ldots P_k = 1/k$

and $P_1^{r_1} \ldots P_{k-1}^{r_{k-1}} P_k^{n_k-1} = 1/k^{n-1}$

and thus

$$\text{Pr \{kth horse wins\}} = \sum_{n=n_k}^{n+1-k} 1/k^n \sum_{r_1=0}^{n_1-1} \cdots \sum_{r_{k-1}=0}^{n_{k-1}-1} \binom{n-1}{r_1, r_2, \ldots r_{k-1}, n_k - 1}$$

In particular, for $n_1 = 5$, $n_2 = 4$, $n_3 = 2$, $k = 3$, $N = 11$

Pr {favorite wins} =

$$\sum_{n=2}^{9} 1/3^n \sum_{r=0}^{4} \sum_{s=u}^{3} \binom{n-1}{r,s,1}$$

$= 15001/19683$.

The other problem yields

$n_1 = 10$, $n_2 = 8$, $n_3 = 6$, $n_4 = 5$, $n_5 = 3$, $n_6 = 2$, $N = 34$, $K = 6$

$$\text{P \{favorite wins\}} = \left(\sum_{n=2}^{29} 1/6^n\right) X$$

$$\sum_{m=0}^{2} \sum_{l=0}^{4} \sum_{k=0}^{5} \sum_{j=0}^{7} \sum_{i=0}^{9} \binom{n-1}{i,j,k,l,m,1}$$

which involves fewer than (29) (2) (4) (5) (7) (9) terms, certainly well within the reach of a reasonable computer.

David Dewan wrote a program to solve this. The job ran 9½ minutes and concluded that the probabilities are .643, .282, .050, .020, .003, and .0004. Off to the races, Mr. Turner.

28 and **29** As usual, anything influenced by Andrew Egendorf (an M.I.T. student colleague) turns out as a disaster. No one responded. Poor George Starkshall should only be criticized for penning his name to an Egendorf creation. Throughout my stay in M.I.T.'s Baker House, Andy would either criticize my pin ball technique (which was both more colorful and productive than his) or berate my *Technology Review* problems as being too boring. It is interesting to note that I was the "star" of the Baker House pin ball column and his problems are the flop of Puzzle Corner.

30 A mathematician moonlighting as a census-taker stops at his friend's house. In this census he is required to obtain the names and ages of all the occupants of the house. After writing down several names and ages the census-taker asks, "Are there any more people who live here?" His friend replies, "Yes, there are three more people that live here." When asked for their ages, the friend reports that the product of the ages is 1296 and the sum is the street number of his house. The census-taker makes a few calculations and then says, "Just tell me one more thing: How many of the three are older than you are?" As soon as his friend replies, the census-taker smiles, writes down the ages, and leaves. What is the house number?

Major Frederick H. Cleveland submitted the following:

"The house number is 91. It is the only sum of the factors of 1296 that occurs more than once (2, 8, 81 and 1, 18, 72); hence the need for more information. Note also that his friend's age must be from 8 to 17 or 72 to 80, inclusive; 1 is a mathematical possibility but not very probable."

Also solved by William Dunbar, Captain Walter C. Moore, Russell A. Nahigian, Smith D. Turner, Captain John Woolson, and Messrs. Martin and Mays.

31 The diagram below shows the final position in a chess game in which White has checkmated Black. What was White's last move? His next-to-the-last move?

Black

						WK	
					WP	BK	
						WP	BP
							WP
WP							
	WB						

White

Captain Martin submits the following: White's next-to-the-last move was P to K5, to which Black responded with P to B4. White then claimed the pawn, giving the published position.

Also solved by: John L. Joseph, T. D. Landale, Alan Matthews, and Donald F. Morrison.

Better Late Than Never

9 Given the quadratic polynomial with matrix coefficients

$$P(Z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + Z \begin{bmatrix} -3 & -1 \\ 14 & -11 \end{bmatrix} + Z^2 \begin{bmatrix} -4 & 4 \\ -58 & 28 \end{bmatrix}$$

Factor it. One solution is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + Z \begin{bmatrix} 2 & -1 \\ 20 & -7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Z + \begin{bmatrix} -5 & 0 \\ -6 & -4 \end{bmatrix}$$

There are five other solutions.

Finally someone solved this. Eric E. Hovemeyer notes that: another factorization of

$$P\ (Z) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + Z \begin{bmatrix} -3 & -1 \\ 14 & -11 \end{bmatrix} + Z^2 \begin{bmatrix} -4 & 4 \\ -58 & 28 \end{bmatrix}$$

is

$$P(Z) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + Z \begin{bmatrix} -4 & 0 \\ -4 & 3 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + Z \begin{bmatrix} 1 & -1 \\ 18 & -8 \end{bmatrix} \right).$$

15 Leslie R. Axelrod claims that 1729 is interesting since it is the first number which is the sum of two cubes in two different ways ($1729 = 1^3 + 12^3 = 9^3 + 10^3$).

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Tech-Crostic

The correct solution to the Tech-Crostic on pages 94 and 95 of this issue of *Technology Review* is as follows:

"Unlike European metalwork which usually utilizes only the fluidity and workability of metals, Japanese metal work often reveals a deep feeling for the structure of metals and their chemical properties."—Smith, *A History of Metallography*.