Puzzle Corner

Allan J. Gottlieb

This summer I purchased a standard "Baker House" (M.I.T.) set of stereo components (A. R. turntable, Dyna amp. and preamp., etc.). What I forgot to realize was that I no longer would be living under standard Baker House conditions. I was so accustomed to having five electrical engineering types within earshot that I didn't consider the problems of maintaining such a stereo "system" on my own.

First came putting the mess together. Well, 6.01 is mostly differential equations and all math types can solve differential equations so I gave it a try. After purchasing a soldering iron and some solder, I opened the box and went to work. First interruption was an emergency visit to the drug store for some burn ointment. After some minor skin surgery, the beasts were assembled. The preamp. didn't work at all. Even the power light didn't go on. At Baker House I would merely ask John Forster, Bob Damis, or Pete Wolfe to help me out. At home I only had my differential equations book, which didn't help at all. After several frustrating hours I found that the line cord was defective.

The amp. worked wonderfully—for two weeks. Then the right channel began to sound awful. Dyna fixed that for a mere $6.50. I could have griped since it was under warranty, but there was an "outside chance" that I soldered it badly. But who cares? I had 120 watts of distortion-free signal going to my cheap speaker from which random sounds erupted. I was really content and was planning to buy new speakers—until the disaster.

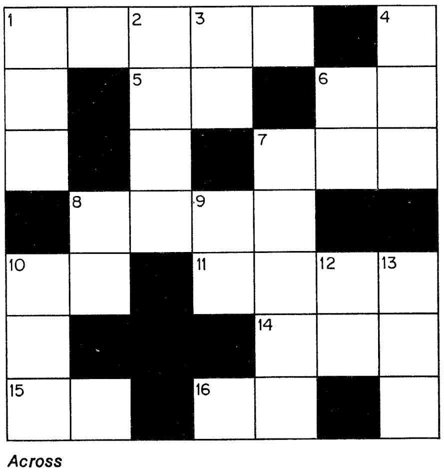
First, the only speaker I had in Boston started to sound funny. It turned out that cheap speakers cannot handle the bass from a good amp. O.K., who wants cheap speakers, anyway? When this one finally blew I brought up my other one and started pricing AR-3's. Then the right channel deteriorated again and was very sensitive to jiggling the input jack. Finally the whole system died.

At least if it were tube equipment I could fry eggs on it. As it is, all my "system" does is sit. The only other Baker House resident around is my colleague Randy Gabel, who knows little more than I do about electrical engineering. He owns a set of headphones which I had hoped to use as test equipment. But now he is afraid to let me even touch them . . . and I cannot really blame him.

Problems

The first problem for this month is from Thomas B. Jabine; it is a famous old English puzzle called Dog's Mead. Mr. Jabine writes:

31 "Although this puzzle relates to a farmer, his family, and his land, it involves a good deal of engineering mathematics and logic. The problem is to find the age of Mrs. Grooby, Farmer Dunk's mother-in-law, and you must not assume the puzzle was invented this year. You'll need to know that there are 20 English shillings to the pound sterling, that an acre is 4,840 square yards, and that a rod is a quarter of an acre. Also, these hints help: One number in the puzzle is the area of Dog's Mead in rods, but it relates to something in the puzzle quite different from that area.



Across

1. Area of Dog's Mead in square yards.
5. Age of Farmer Dunk's daughter, Martha.
6. The difference between the length and breadth of Dog's Mead in yards.
7. Number of rods in Dog's Mead times number nine down.
8. The year when Little Piggly came into occupation by the Dunk family.
10. Farmer Dunk's age.
11. The year Farmer Dunk's youngest child, Mary, was born.
14. Perimeter of Dog's Mead in yards.
15. The cube of Farmer Dunk's walking speed in miles per hour.
16. Number fifteen across minus number nine down.

Down

1. The value of Dog's Mead in shillings per acre.
2. The square of Mrs. Grooby's age.
3. The age of Mary.
4. The value of Dog's Mead in pounds sterling.
6. The age of Farmer Dunk's first-born, Edward, who will be twice as old as Mary next year.
7. The square, in yards, of the breadth of Dog's Mead.
8. The number of minutes Farmer Dunk needs to walk one and one-third times around Dog's Mead.
9. See number ten down.
10. Ten across times nine down.
12. One more than the sum of the digits in the second column down.
13. Length of tenure, in years, of Little Piggly by the Dunk family.

The next problem is from Frank Rubin:

32 Let $p_1, p_2, \ldots p_n$ be points in the plane such that distance $(p_i, p_j) \leq 1$ for $1 \leq i \leq j \leq n$. Prove that these points lie within a circle of radius $1/3\sqrt{3}$.

Francis A. Packer, Jr., sends the next one:

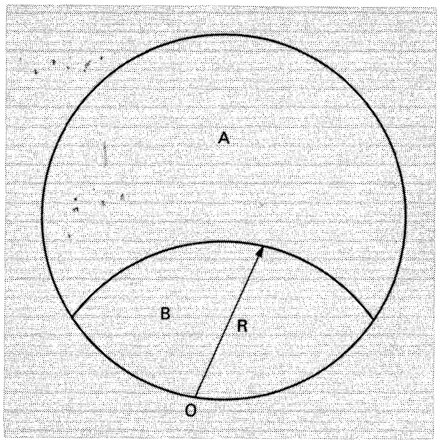
33 Construct a triangle given the three altitudes.

The following was contributed by Cornel Somogy of the Association of American Railroads Research Center:

34 a and A are the surface areas, v and V the volumes of a smaller and a larger sphere, respectively. If $A = (a + 10)$ square inches and $V = (v + 10)$ cubic inches, what are the corresponding radii? Solve to two decimals!

H. Weber Hartmann had a dilemma. He writes: "I have had a simple problem harrassing me for quite some time and hesitated to ask anyone to solve it for me due to its ridiculousness. But upon reading the January, 1969, issue of *Technology Review* I came across the corrected solution of the grazing cow problem and decided mine wasn't too silly at that. So here goes:

35 "A cow is grazing in a circular field A of any given area, say 10 acres. She is tied at 0 with a chain R long. How long must the chain be for her to cover an area B of one acre?"



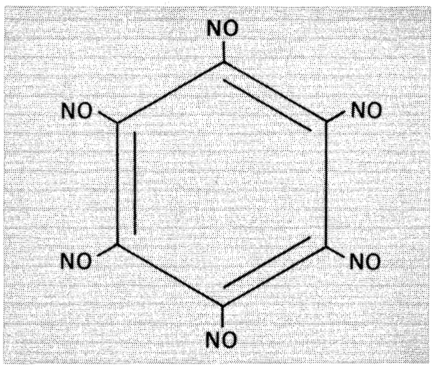
Speed Department

Tom Amaier submits the following:
"I enjoyed seeing a 'chemical' problem in your Speed Department in February so decided to give you a few more. The one you gave requires a knowledge of chemistry (Fe^{++} = ferrous); but mine require even less knowledge of chemistry:

SD14 H I O Ag—a famous saying.

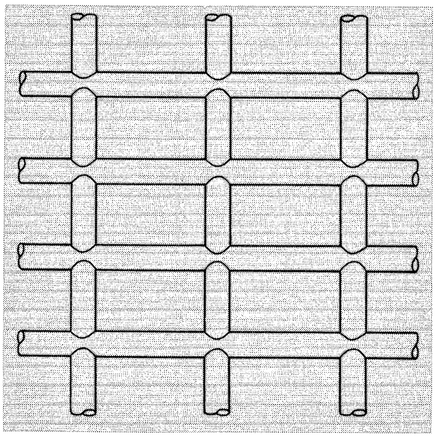
SD15 Ba $(Na)_2$—food.

SD16 A new drug

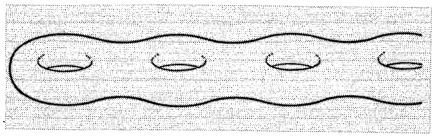


Solutions

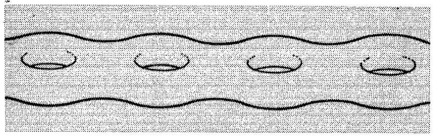
16 Consider the surface of an infinite jail cell, which extends up, down, left, and right:



And two infinite holed tori, one extending to the right:



. . . and one extending to the left and the right:



Are any two of these three homeomorphic? Why, or why not?

No one solved this problem, so here's a hint: the jail cell and the double infinite torus *are* homeomorphic but the single infinite torus is different.

17 If an animal is tethered to one side of a circular silo in an open field by a rope with length equal to the circumference of the silo and can graze over an area of one acre, what is the outer diameter of that silo?

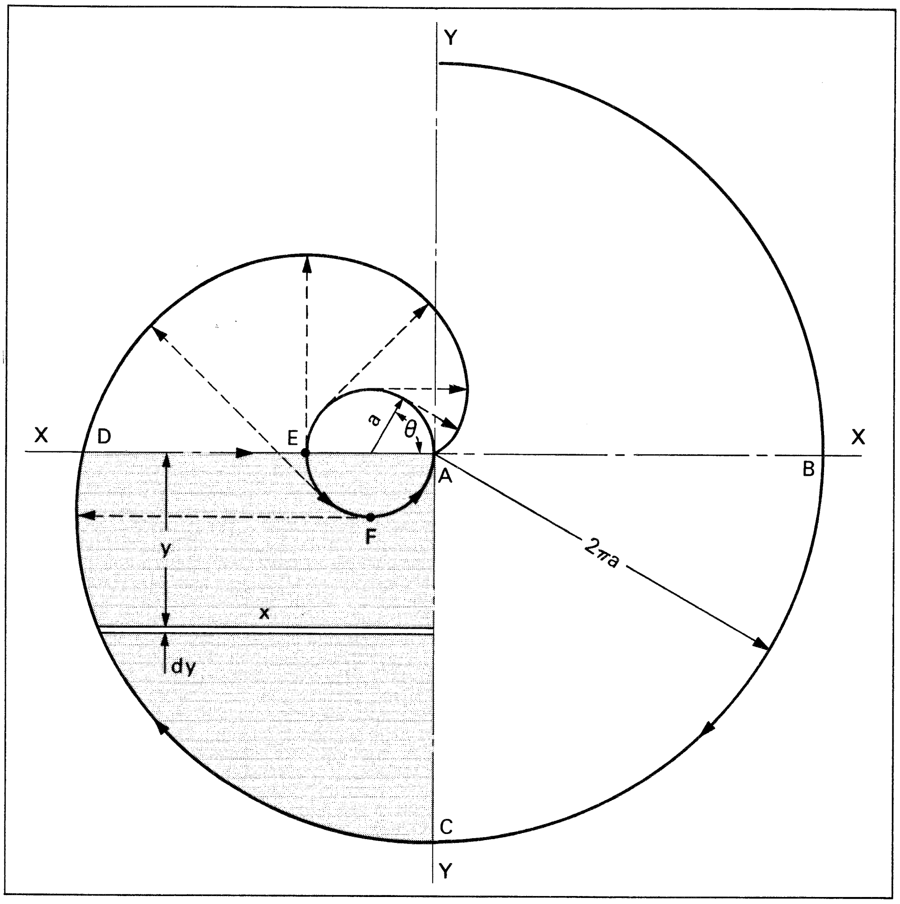
The following solution is from Karl E. Schoenherr:

Referring to the sketch, let a be the unknown radius of the silo and A be the point at which the rope tethering the animal is attached. The length of the rope is then $2\pi a$. Construct an X—Y system of coordinates with the origin at A, and draw the involute of the circle representing the silo as shown in the figure.

Assume that the animal starts to graze at A and proceeds to B along the X axis; then, holding the rope tight it proceeds along the circular arc from B to C and then along the arc of the involute from C to D. From there the animal proceeds along the X axis to E and around the circumference of the silo back to A. The area enclosed by these boundaries is one half of the maximum grazing area reachable by the animal, and therefore is to be one-half acre by hypothesis. If this area is denoted by G, we have

$$G = \tfrac{1}{4}(2\pi a)^2\pi - \tfrac{1}{2}(a^2\pi) + \int_0^{2\pi a} x\,dy \quad (1)$$

where the integral is the area shown



shaded in the drawing. The parametric equations of the involute of the circle with respect to the chosen coordinate system are

$x = a(\cos\theta + \theta\sin\theta) - a$ (2)

$y = a(\sin\theta - \theta\cos\theta)$ (3)

where θ is the angle between the radius vector a and the positive direction of the X axis, as shown. Differentiating (3), we get

$dy = a\,\theta\sin\theta\,d\theta.$ (4)

Therefore, the area under the involute arc is

$$G_i = \int_0^{2\pi a} x\,dy =$$

$$a^2\int_\alpha^\beta (\cos\theta + \theta\sin\theta - 1)\,\theta\sin\theta\,d\theta$$

$$= a^2\left[\int_\alpha^\beta \theta\sin\theta\cos\theta\,d\theta + \int_\alpha^\beta \theta^2\sin^2\theta\,d\theta - \int_\alpha^\beta \theta\sin\theta\,d\theta\right]$$

$= I_1 + I_2 + I_3$ (5)

The limits β and α have the following values:

$\beta = 2\pi$ (6)

$\alpha = 1.4274\pi$ for which $y = 0$. Evaluating the integrals, we get

$I_1 = \frac{1}{2}\left[\frac{1}{4}(\sin 2\theta) - \frac{1}{2}(\theta\cos 2\theta)\right]_\alpha^\beta$ (7)

$I_2 = \frac{1}{4}\left[\theta\sin(2\theta\sin\theta - 2\theta^2\cos\theta) + 2/3(\theta^3) - \theta - 1/2(\sin 2\theta\right]_\alpha^\beta$ (8)

$I_3 = \left[\sin\theta - \theta\cos\theta\right]_\alpha^\beta$ (9)

Therefore,

$G_i = a^2\left[\frac{1}{4}(\sin 2\theta) - \frac{1}{4}(\cos 2\theta) - \sin\theta + \theta\cos\theta + \frac{1}{4}[\theta\sin(2\theta\sin\theta - 2\theta^2\cos\theta)] + 1/6(\theta^3) - \theta/4\right]_\alpha^\beta.$ (10)

Introducing the limits from (6), we have

$G_i = 44.8977\,a^2 - 15.2003\,a^2$

$= 29.6974\,a^2.$

The area of the quarter circle of radius $2\pi a$ minus the area of the half circle representing the silo is

$G_c = 29.4355\,a^2.$

Therefore,

$G = (29.4355 + 29.6974)\,a^2$

$= 59.1329\,a^2.$

Setting this area equal to one-half acre, we get

$21{,}780 = 59.1329\,a^2$, whence

$a = \sqrt{21780/59.1329} = 19.1918$ feet.

Also solved by Tom Maier, Richard T. Roca, Douglas J. Hoylman, George Todd, John E. Prussing, R. Robinson Rowe, Mark Yu, K. B. Blake, and Lyndon S. Tracy.

18 It is always possible to find arbitrarily long sequences of consecutive composite numbers. Suppose, for example, that we wish to find five consecutive composite numbers. We define $6! = 1\cdot2\cdot3\cdot4\cdot5\cdot6 = 720$. Then numbers

$6! + 2 = 722$

$6! + 3 = 723$

$6! + 4 = 724$

$6! + 5 = 725$

$6! + 6 = 726$

are all composite; 722 is divisible by 2, 723 by 3, 724 by 4, and so on.

Is this the first time in the sequence of natural numbers that there are five consecutive composite numbers?

Captain John Woolston, U.S.N., submits the following:

"The answer is, by inspection, 'Hell no!' as any elementary school child could tell you. Assuming a narrow definition of consecutive composite numbers as illustrated by the problem, i.e., consecutive numbers divisible by consecutive integers, the requirement for $(N + 2)/2$, $(N + 3)/3, \ldots (N + M)/M$ to be integers is merely that $N/2, N/3, \ldots N/M$ be integers. The elementary school child, unconfused as well as unenlightened by higher mathematics, faces this in his early years in finding the lowest common denominator, so obviously 5! meets the test (since 2 x 3 = 6). So also does 3 x 4 x 5 or 5!/2!, i.e., 120 and 60. So also any integer multiple of 60 meets the test.

"Of course, the use of 12 and 60 as number bases in mathematics is rather old. Babylonians among others probably used them for just this reason, since I doubt if the Babylonians were terribly bothered by polydactylism. Of course, in the broader definition of consecutive numbers, i.e., not prime, the first series of five is the series 24, 25, 26, 27, 28; but here I must lean on individual checking rather than on neat mathematic solutions to find the distribution of primes and the length of the gaps between them."

Also solved by Lyndon S. Tracy, James Shearer, G. J. Todd, Ernest W. Thiele, Tom Amaier, and Messrs. Rowe, Prussing, and Hoylman.

19 Neither side vulnerable:

North
♠ Q J 7 3 2
♥ K 6 5
♦ A 6
♣ 10 8 2

West
♠ 9 8 6 5
♥ J 9 7
♦ K
♣ J 9 7 6 5

East
♠ K 10 4
♥ A 3 2
♦ 7 5 4 3 2
♣ Q 3

South
♠ A
♥ Q 10 8 4
♦ Q J 10 9 8
♣ A K 4

The bidding:

North	*East*	*South*	*West*
Pass	Pass	1 ♦	Pass
1 ♠	Pass	2 ♥	Pass
3 ♥	Pass	4 ♥	Pass
Pass	Pass		

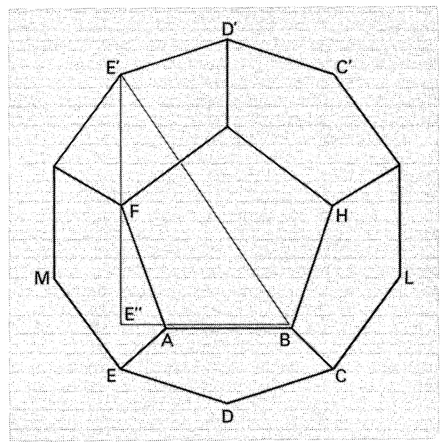
The opening lead: ♠ 5.

No solutions have been received for this problem. Here is how the M.I.T. team, originally playing the hand in a Cavendish Club tournament, brought it off:

By forcing the opponents to use their trumps, declarer was able to maintain a tight hold over a delicate contract. The opening lead went to East's ♠ 10 and South's ♠ A. South played a diamond, capturing West's ♦ K with dummy's ♦ A. A diamond return was ruffed by West's ♥ 7. West shortened declarer's trump holding by a second spade to dummy's ♠ J and East's ♠ K. Another diamond was ruffed by the ♥ 9 and overruffed by the ♥ K. The SQ permitted the declarer to discard a club. A club to the ♣ A and ♣ K was followed by another diamond, permitting the defense to win their ♥ J and ♥ A at their leisure, thereby limiting them to three trump tricks.

20 Express the volume of a regular dodecahedron in terms of the length of an edge—without the use of trigonometry.

Smith D. Turner was the only one to solve this; his solution follows:
The volume is that of 12 pyramids:



Thus the volume is 12 x 1/3 x area of one face x radius of the inscribed sphere (R). The last term is difficult. On the sketch, let E″ be the projection of E′ on the plane of the base ABC. Then $(2R)^2 = (E'E'')^2 = (E'B)^2 - (E''B)^2$.
To obtain E′B, pass a plane through FHL, etc., and as the sides of the pentagon formed are diagonals of the faces, *diagonals* of this larger pentagon (e.g., MH) $= e(3 + \sqrt{5})/2 = E'B$. $E''B = 2 \times$ radius of circle inscribed in face

$= e\sqrt{(5 + 2\sqrt{5})/20}.$

Substituting these:

$2R = e\sqrt{[(3 + \sqrt{5})/2]^2 - [\sqrt{(5 + 2\sqrt{5})/20}]^2}$

$R = e\sqrt{(25 + 11/5)/40}.$

So $V = 12 \times 1/3 \times 5/2$

$\left(e^2\sqrt{(5 + 2\sqrt{5})/20}\right.$

$\times\, e\sqrt{(25 + 11\sqrt{5})/40}$

$= 5\,e^3\sqrt{(47 + 21\sqrt{5})/40}$

$= e^3(15 + 7\sqrt{5})/4 = 7.6631189\,e^3$

Better Late Than Never

1 Frank G. Smith.
10 Norman L. Appollonio and John P. Vint.
12 R. E. and M. K. Bhananaskhian, Richard Lopes, and Mr. Amaier.
15 Mr. Lopes.

Mr. Gottlieb, who graduated from M.I.T. in mathematics in 1967, is a teaching assistant at Brandeis University. Send answers and problems to him at the Department of Mathematics, Brandeis University, Waltham, Mass., 02154.