Puzzle Corner

Allan J. Gottlieb

Hi. My backlog of puzzles is finally starting to dwindle. Hence there should not be any one-year delays between receipt of problems and their subsequent appearance in the column. In fact, if a few new ones aren't received within a few months, I will have to use some of my own. And since this invariably leads to degradation of the column, we would appreciate some fresh outside contributions. Bridge problems are in critically short supply and are particularly welcome.

Problems

The first problem this month is from John Warner Leech of Arlington, Mass.:

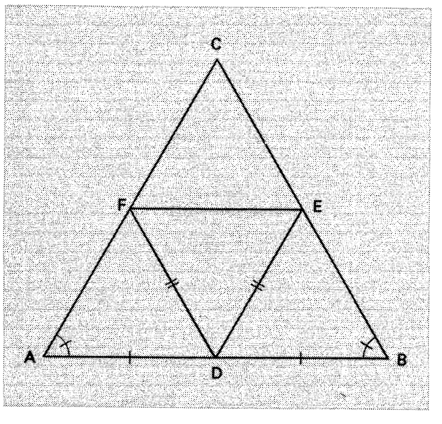
21 One commuter to M.I.T. cannot use regular gasoline in his car because the engine knocks. He does not wish to use premium gasoline exclusively because of the expense. So he uses the following scheme to eliminate the knock and reduce the expense: Each time the gas tank is half full he fills it, alternating between regular fuel (92 octane) and premium fuel (100 octane). Assuming that he has been doing this for some time, what is the octane level of the fuel in the tank after he has filled it with regular fuel? (Assume that 10 gallons of 92-octane fuel plus 10 gallons of 100-octane fuel yields fuel of 96 octane.)

Since most people find the geometry problems the easiest, I have decided to print one which appears to me to be rather formidable. Should the wording sound a little strange, bear in mind that it comes from Mr. Fine of Gloucester, England; he admits that it may be "too advanced" for the *Review!* And he also says that "though it is fairly easy to find special cases (e.g., a recurrent sequence), I have not really 'got my teeth' into this problem—which is well over 20 years old."

22 If a pair of triangles is not co-polar, the joins of corresponding vertices form a triangle and so do the intersections of corresponding sides. The original pair of triangles has been transformed into a second pair which can be transformed into a third and so on. How does the sequence of pairs of triangles behave?

The following problem arose from a study of the "bible" (at least during my freshman year at M.I.T., this was the "bible"). William J. Wagner of San Carlos, Calif., writes:

23 In a high school calculus class I assigned a problem from George B. Thomas' text which asked for the maximum area of an isosceles triangle DEF whose vertex D is the midpoint of the base of an isosceles triangle ABD. The solution requires the knowledge that FE is parallel to AB. Without giving it much thought I started to sketch a proof of this fact for the class but quickly realized I wasn't getting anywhere. Two days later I figured it out. Can you prove that FE is parallel to AB?



John E. Prussing of La Jolla, Calif., has a question about some infinite series. One clue as to how fast mathematics is changing is that, although I graduated only five years after he did, I find some of his notation unfamiliar. Hence I will accept, for partial credit, the mere definition of the function signum (x).

24 For the case $n = 10$, $x = \frac{1}{2}$, analytically evaluate the infinite series:

$$\sum_{k=1}^{\infty} [a_k k(k+1) + x^k [k - (-1)^k]/k]$$

where

$a_k = \frac{1}{2}\,[1 + \text{signum}\,(n - k + \epsilon)]$,
$0 < \epsilon < 1$.

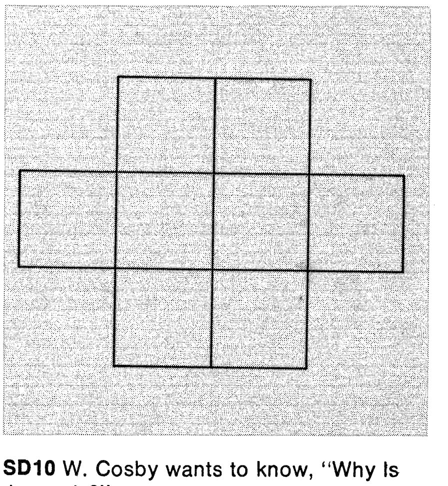
Here is a cool number theory problem from Eric E. Hovemeyer:

25 Let N be the number of positive integers which contain no digit more than once when they are expressed in base b where b is an integer greater than 2. Show that N is always composite. Example: When $b = 3$, $N = 10$, since there exist ten positive integers which contain no digit more than once when they are expressed in base 3. Expressed in base 3 they are 1, 2, 10, 12, 20, 21, 102, 120, 201, 210.

Speed Department

From Marshall Greenspan of Fairfield, Conn.:

SD9 Arrange the digits 1 through 8 in the boxes of the figure below such that no two consecutive digits are adjacent either vertically, horizontally, or diagonally.



SD10 W. Cosby wants to know, "Why Is there air?"

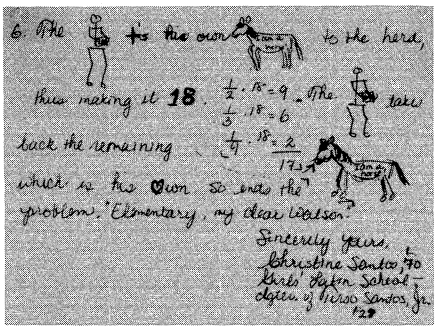
Solutions

6 A man dies leaving a will stating that one-half of his horses go to the eldest son, one-third to the next oldest, and one-ninth to the youngest. Alas, at the time of death the man has 17 horses. However, a shrewd lawyer solves the problem without killing any of the horses. How?

The following is from Paul Karger, an M.I.T. undergraduate: "The honest upstanding Techman-turned-lawyer would simply add one of his own horses, making 18. One-half of 18 is 9. One-third of 18 is 6, and one-ninth of 18 is 2. This leaves one horse which the Techman takes home. However, the true lawyer would divide 17 by 2 giving 8½. Truncating the fraction, the first son gets 8 horses. Likewise, the second and third sons get 5 and 1 horses, respectively. Two horses go for inheritance tax, and the lawyer keeps one for his fee."

Also solved by Joseph Adolph, Richard A. Bator, Brian Bucallo, William S. Dunbar (Trinity Pauling School), Mr. Greenspan, Mary Lindenberg, Russel A. Nahigian (the proposer), Mr. Prussing, Ralph Segal, John D. Sigel (Derryfield School), Neil Steinmetz. Smith D. Turner (who signs his name $\int dt$), and Captain John Woolston.

For those of you who prefer the *Daily News* (New York's picture newspaper) approach, the following should be more informative:



7 Consider an infinite chessboard having one edge 8 squares long and all 32 pieces. What is the largest number of pieces which can be on the board and there still be no legal moves for either side?

Two different interpretations of this question have been submitted (notice the difference in their respective boards). The first is from Donald Oestreicher of Cambridge:

	Pawn					Pawn	
	Pawn	Pawn	Pawn	Pawn	Pawn	Pawn	
Knight	Rook	Bishop	Queen	King	Bishop	Rook	Knight
Edge of the other side of the board							

The other is from Mark Yu, an M.I.T. undergraduate, who writes: "This sort of problem really deserves a hedgy answer, like 31 pieces (one king missing) since no moves are allowed when the game is over. I could only manage to get 26 pieces on the board (side 1 moves down):

		B_1	R_1	R_1	B_1		
	P_1		P_1	K_1		P_1	
	P_2		P_1			P_2	
			P_2			P_2	
			P_1			P_1	
	P_1		P_2			P_1	
	P_2		P_2	K_2		P_2	
		B_2	R_2	R_2	B_2		

"It is interesting to note that if one of the side edges is fixed, 32 pieces may still be placed on the board:

N_1	R_1	N_1	B_1				
Q_1	R_1	K_1		P_1			
B_1	P_1		P_1	P_2	P_1		
	P_1		P_2		P_2		
	P_2		P_1		P_1		
B_2	P_2		P_2	P_1	P_2		
Q_2	R_2	K_2		P_2			
N_2	R_2	N_2	B_2				

(pawns 1 move down)."

8 and **9** These problems, which appear so wildly different, actually have something in common. Nobody has solved either one!

10 A cylindrical hole of length 6 inches is drilled through the center of an ivory ball. What is the volume of the ivory remaining after the hole is drilled?

Mr. Greenspan had little trouble with this one:
"Since the radius of the cylindrical hole is not given, the remaining volume of the sphere must be independent of the cylinder radius. By setting this radius equal to zero, the diameter of the sphere becomes 6 inches and thus the volume is given by:

$V = [4\pi(6/2)^3]/3 = 36\pi$ cubic inches.

This result can, of course, also be shown analytically by assuming the sphere has a radius of $(3 + h)$ and thus the cylindrical hole has a radius of $[(3 + h)^2 - 3^2]^{1/2}$ or $(6h + h^2)^{1/2}$

The cutout section consists of a 6-inch long right circular cylinder whose volume is
$V_1 = 6\pi(6h + h^2)$

plus two spherical segments each with volume
$V_2 = 2/3\pi h^3 + 3\pi h^2$.

Therefore the total volume removed from the sphere is:
$V_T = V_1 + 2V_2 = 36\pi h + 12\pi h^2 + 4/3\pi h^3$.

The volume of the original sphere was
$V_3 = 4/3\pi(3+h)^3 = 36\pi + 36\pi h + 12\pi h^2 + 4/3\pi h^3$.

Thus, the remaining volume of the sphere is
$V_R = V_3 - V_T = 36\pi$."

Also solved by Norman L. Apollonio, James Barton, T. E. Dadson, Mr. Prussing, J. J. Shipman, Jack Teller, Mr. Turner, James Weigl, Mr. Yu, and Andrew Zeger.

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