Puzzle Corner

Allan J. Gottlieb

Since this issue contains solutions to problems published in October/November, I am keeping my introductory yak-yak to a minimum.

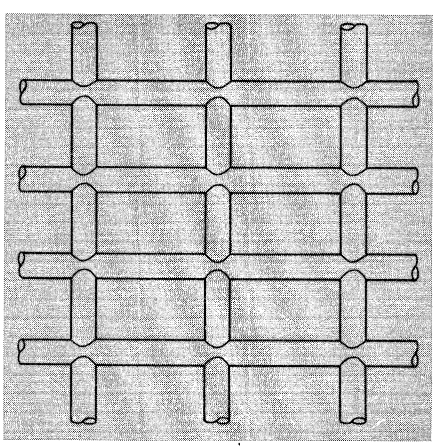
But I want to renew my plea that everyone mention the problem number with every comment or answer. And I ask that we finally put to rest all of last year's (Volume 70) puzzles—even the bridge problems!

Problems

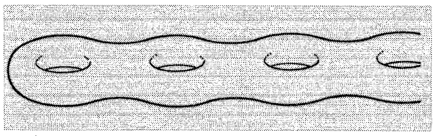
Recently I met a few of my colleagues from M.I.T. They mentioned that they read the column but are sorry to see that no "mathy" problems appear. Of course, they had none to submit, but I will respond with the following challenge.

The following was explained to me by Mike Spivac (of *Calculus on Manifolds* and *Calculus Calculus Calculus Calculus Calculus Calculus Calculus Calculus* fame). It concerns the classification of noncompact, nonbounded 2-manifolds.

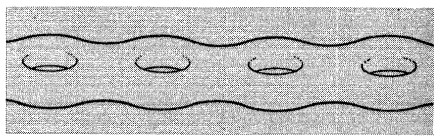
16 Consider the surface of an infinite jail cell, which extends up, down, left, and right:



And two infinite holed tori, one extending to the right:



. . . and one extending to the left and the right:



Are any two of these three homeomorphic? Why, or why not?

Frank G. Smith has another grazing problem. He writes, "That 'grazing around a square barn' puzzle was not so bad. Even I could solve it. Now here is one I can state but cannot solve. I've forgotten types and equations of symmetrical curves and also my integral calculus. So I leave this one for the 'math boys' to solve. And don't give it to that one who solved the square barn so easily and forgot the animal could graze clockwise *and* anticlockwise! Here is the problem:

17 If an animal is tethered to one side of a circular silo in an open field by a rope with length equal to the circumference of the silo and can graze over an area of one acre, what is the outer diameter of that silo?"

The next problem, from R. Wells Johnson, is reprinted from the Bowdoin College magazine:

18 It is always possible to find arbitrarily long sequences of consecutive composite numbers. Suppose, for example, that we wish to find five consecutive composite numbers. We define $6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$. Then numbers
$6! + 2 = 722$
$6! + 3 = 723$
$6! + 4 = 724$
$6! + 5 = 725$
$6! + 6 = 726$
are all composite; 722 is divisible by 2, 723 by 3, 724 by 4, and so on. Is this the first time in the sequence of natural numbers that there are five consecutive composite numbers?

Here is what everyone has been waiting for—another bridge problem! It is "for real," as described last year by Mel Creem in the Boston Globe: "The latest series of I.M.P. scored pair events resulted in victories for Owen and Phyllis Rye at the Chess Club and for M.I.T. Students Bob Cohen and Marty Levin at the Cavendish Club. We have noted the high caliber of play of many of the young M.I.T. players. For example, in a set match with Dick Freedman and Ken Lebensold against tournament veterans Les Popper and Norman Humer, Lebensold, as declarer, made a singular contribution to the M.I.T. victory in this deal:"

19 Neither side vulnerable:

North
♠ Q J 7 3 2
♥ K 6 5
♦ A 6
♣ 10 8 2

West
♠ 9 8 6 5
♥ J 9 7
♦ K
♣ J 9 7 6 5

East
♠ K 10 4
♥ A 3 2
♦ 7 5 4 3 2
♣ Q 3

South
♠ A
♥ Q 10 8 4
♦ Q J 10 9 8
♣ A K 4

The bidding:

North	*East*	*South*	*West*
Pass	Pass	1 ♦	Pass
1 ♠	Pass	2 ♥	Pass
3 ♥	Pass	4 ♥	Pass
Pass	Pass		

The opening lead: ♠ 5.

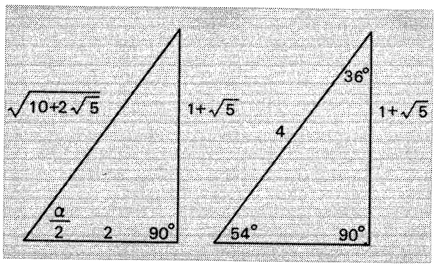
In examining a problem from last year, Robert D. Scott finds a subtler one for this year. He writes:

"Mr. Hovemeyer in your July/August, 1968, issue of *Technology Review* makes his point well regarding problem 5 in the October/November, 1967, issue. (The original problem: express the volume of a regular dodecahedron in terms of the length of an edge.—Ed.) I also find it disconcerting to see such a poor approximation given for such an elegant principle as proposed by Mr. Severn.

"Having previously cut a few regular dodecahedrons in crystal and wood. I became interested in solving the problem myself, as Mr. Severn's terms in the printed solution were not too easy to understand. Doing so I find there is a lot of magic in the dodecahedron and suspect I've only found part of it.

"My first solution was:
$V = 5\ a^3 \tan^2 54° \sin 54°$
which seems more tolerable than Mr. Hovemeyer's solution and oddly enough depends on the unusual facts that
$2(\sin 54°) = \sqrt{1 + 2(\sin 54°)}$
$= \tan \alpha/2 = (1 + \sqrt{5})/2$
$= \tan (\sin^{-1} [1/(2 \sin \pi/5)])$
$= \tan [(\pi - \tan^{-1} 2)/2]$
where $\alpha = 116° 34'$, the dihedral angle of the regular dodecahedron.

"The following two basic triangles of the dodecahedron are in effect the route to an even less complex solution:



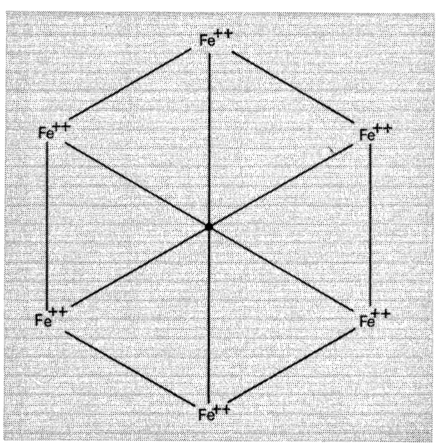
Using these relationships, then, my final solution is
$V = [(5 + 5\sqrt{5} + 5)/(5 - \sqrt{5})]\ a^3$
$= 7.663119\ a^3$,
which probably means that there is another puzzle for you:

"**20** Solve the original problem—express the volume of a regular dodecahedron in terms of the length of an edge—without the use of trigonometry. (Is this mystical preponderance of fives due to the pentagonal basis of the dodecahedron?)"

Speed Department

SD8 How come the year in which the British constitution was written has no prime factor?

SD9 What is this?



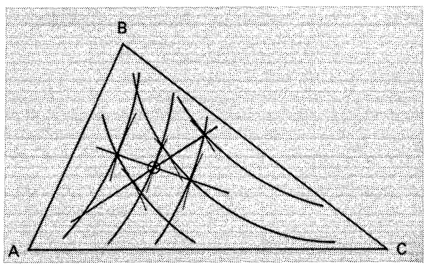
Solutions

The following solutions are to problems which were published in the October/November issue of *Technology Review* (p. 92). Readers are invited to contribute solutions to the problems first published (above) in this issue of the *Review;* they will be printed in the third succeeding issue (May).

1 Given an arbitrary triangle, find (by geometrical construction) the point such that the sum of the distances to the three vertices is a minimum.

Many fine solutions were submitted. Some relied on physics, others on calculus. But only the proposer did it geometrically, as desired:

"Consider the triangle:



Construct a family of ellipses having A and B as focii. The sum of the distances to A and B at any point on a given ellipse will be constant. Now for each ellipse constructed, determine the point on that ellipse such that the distance to C is a minimum (this can be done by finding the point of tangency of a circle drawn from C). This will yield a series of points through which a curve may be drawn (the curve DE, above). This procedure can be repeated for points B and C as focii and another curve drawn with A as the vertex to which distances must be a minimum. Since the point of minimum distance must be located on both DE and FG, the intersection of both curves is the required point."

Solutions also came from Russell L. Mallett, Jules Sandock, Michael Goldberg, W. Allen Smith, Glenn Stoops, F. T. Leahy, Jr., and Mark Yu (whose solution was geometric—but complicated).

2 Find all integral solutions to
$x^2 - 8xy - 2y^2 - 6x + 1 = 0$,
noting the possibility that none exists.

Mr. Stoops noticed that
$x^2 - 8xy - 2y^2 - 6x + 1$
$= (3x + 1)^2 - 2(x - y)^2$,
so any integer solution would show that $\sqrt{2}$ is rational.

Also solved by John E. Prussing, Guy F. Boucher. Alexander Bogan, Jr., P. W. Parsons, Eric Rosenthal, Douglas Hoylman, K. B. Blake, Russell L. Mallett, Norman M. Wickstrand, Jeffrey S. Passel, Harry Simon, and Messrs. Yu and Smith.

3 Consider the series

$$\sum_{n=1}^{\infty} \sin (n!\pi x).$$

a. Show that it converges when x is rational.
b. Show that it converges when $x = e$.
c. Find a value of x for which the series diverges.

Once again the best response was by the proposer, in this case Mr. Hoylman:
a. Let $x = p/q$, p and q integers. Then if $n \geq q$, n! contains q as a factor, so $n!x$ is an integer. The sine of an integral multiple of π is zero, so in this case the series contains only finitely many nonzero terms.
b. $e = 1 + 1 + 1/2! + 1/3! \ldots$
$+ 1/(n - 2)! + 1/(n - 1)! + 1/n!$
$+ 1/(n + 1)! + 1/(n + 2)! + \ldots$
$n!e = n! + n! + n(n - 1) \ldots 3 + \ldots$
$+ n(n - 1) + n + 1 + 1/(n + 1)$
$+ 1/[(n + 1)(n + 2)] + \ldots$
Let $I_n = N! + n! + \ldots + n + 1$
and $r_n = 1/(n + 1) +$
$1/[(n + 1)(n + 2)] + \ldots$
Then I_n is an integer of the same parity as $n + 1$ (since every other term has $n(n - 1)$ as a factor and hence is even, and
$1/(n + 1) < r_n < 1/(n + 1)\ 1/(n + 1)^2$
$+ 1/(n + 1)^3 + \ldots = 1/n$.
So $\sin (n!\pi e) = \sin (I_n\pi + r_n\pi)$
$= (-1)^{n+1} \sin (r_n\pi)$
and $\sin \pi/n > \sin (r_n\pi) > \sin \pi/(n + 1)$
$> \sin (r_{n+1}\pi) > \ldots$
since sin is an increasing function. So we have an alternating series in which the terms are decreasing in absolute value and tending to zero (since sin is continuous and $\sin 0 = 0$). Hence it converges.
c. Let $x = 2\ e$. Then, with the same notation as above,
$\sin (2\ en!\pi) = \sin (2\ I_n\pi + 2\ r_n\pi) =$
$\sin (2\ r_n\pi) > \sin 2\ \pi/(n + 1)$
$> \pi/(n + 1)$
for sufficiently large n (since $\lim_{x \to 0} (\sin x)/x = 1$, hence $(\sin x)/x > 1/2$ for x sufficiently close to zero). Hence this series dominates a harmonic series, so it diverges.

Also solved by D. Thomas Zerwilliger and Messrs. Prussing and Stoops.

4 No takers as yet. Keep at it!

5 I can't figure out cryptograms or their solutions, so I shall simply reprint a solution from B. M. Rothleder without making any claims at understanding: "The key word is the author's name; the crypt is:

A B C D E F G H I J K L M N
D B S I L V E R M A N C F G
O P Q R S T U V W X Y Z
H J K O P Q T U W X Y Z

The solution is: 'A non plus clue not used MIT mag so wild guess author HOJO thus key to crypt is to know platitudes stop DBS.' " (Mr. Rothleder notes errors in the italicized letters.—Ed.)

Also solved by Robert Sinnott.

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