Puzzle Review

Allan J. Gottlieb, '67

Now that doctoral qualifyings are finished and my master's degree secured, I can breathe easily for a while. It is hard to believe, but those quals were the last exams of my career. Hopefully the let-up of pressure will result in my writing a better column.

Due to the wonderful response I am receiving, there is an ever-increasing backlog of proposed problems. We are considering running eight or nine of them each month, instead of this year's five. The only trouble is space. Perhaps we will offer many puzzles and for some print only the (numerical) answers, leaving out all computations. If you have any suggestions, why not send them in? My address, as always, is at the Department of Mathematics, Brandeis University, Waltham, Mass., 02154.

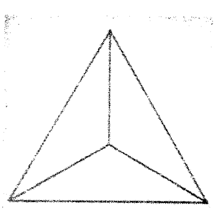
In order to avoid a six-month turn-around time for solutions, I shall propose no new problems this month but, instead, will answer problems proposed in both April and May. The June solutions will be published in the first fall issue (October/November).

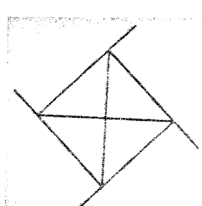
Solutions

25 Given a rectangular solid box with inside dimensions of 18"×51"×69" and incompressible golf balls 1.82" in diameter, find the maximum number of golf balls that can be packed into the box. Assume that gravity is present and that the box may be rotated to any spacial orientation, so long as it is not deformed.

I am afraid we have no takers. Since some interest has been shown, a solution would be appreciated. (Eric E. Hovemeyer says at least 11,100 are needed.)

26 Given six matches of equal length, make four identical triangles without breaking the matches.





The usual solution, proposed by Mr. Hovemeyer, Harry V. Ellis, 3d, '65, A. Glenn Stith, '64, and Mark Yu, '70, is a regular tetrahedron (below left); Richard P. Bishop, '59, offers a different construction (below right):

27 Find the solution for the case of two odd balls both defective by the same absolute amount (but they may be opposite in sign).

Somehow, only half the problem (as above) was printed. Despite this handicap Mr. Yu attempted the problem; unfortunately I cannot understand his solution, and indeed I cannot even understand the problem.

28 Using a two-pan balance, given a set of n weights, each of integral weight $w_1, w_2, \ldots w_n$ such that any object with an integral weight from unity to $w_1 + w_2 + \ldots w_n$, the sum of all weights can be determined.

The most complete solution is from Martin J. Krone, '66:
(1) If we are to use only one side of the pan balance, then the binary weights 1, 2, 4, 8, 12, . . . will work uniquely.
(2) If we know *a priori* that only integer weights will occur, there is no need to cover every integer; the even integers will do, for example. Thus 2, 4, 8, 16 . . . will suffice as weights.
(3) If we are allowed to put known weights on both sides of the balance (subtract weights, in effect), then the problem is much more interesting. My reasoning is as follows:

a. We need only cover even weights.
b. The weights w_n should be chosen as follows: suppose we have N weights whose sum is S_N, and we are allowed to use both sides of the balance. Then w_{n+1} should be chosen so that

$$w_{n+1} - S_N = S_N + 2$$

$$w_{N+1} = 2 + 2S_N$$

assuming that the weights $w_1, w_2, \ldots w_N$ are capable of balancing all even weights from 2 to their sum. (This formula is arrived at by requiring no redundancy in the combinations.) The first few weights by this method are: $w_1 = 2$, $w_2 = 6$, $w_3 = 18$, $w_4 = 54$, and $w_5 = 162$. The following difference equation then results:

$$S_{N+1} = 2 + 3S_N.$$

The solution by standard methods is

$$S_N = 3^N - 1, \text{ and}$$

$$W_N = 2 \times 3^N.$$

Thus for solution (3) the determinable set of weights increases as 3^N; this is in contrast to $2^N - 1$ and $2^{N+1} - 1$ for solutions (1) and (2), respectively. I believe that solution (3) is optimal (i.e., the determinable set S_N increases fastest with the number of weights N) but can't prove it, being a lowly engineer (sic).

Also solved by Mr. Bishop and Leon Sutton, '62.

29 With the following, South is the declarer at seven spades. West leads ♣8. The problem is to make seven against any defense.

	♠ K J 9 7 6 ♥ A J 9 6 5 ♦ — ♣ A J 9	
♠ Q 10 8 ♥ K ♦ K 3 2 ♣ 8 7 6 5 4 3		♠ — ♥ Q 10 8 7 4 3 2 ♦ Q 10 8 7 4 ♣ K
	♠ A 5 4 3 2 ♥ — ♦ A J 9 6 5 ♣ Q 10 2	

I like the following by F. Ted Leahy, '33: Win ♣ A, win with ♣10, play low ♠, win with ♣Q, and play low ♠ again, always winning as cheaply as possible in dummy. Play highest ♠ remaining in dummy, thus pulling West's last trump. During these six tricks, East has had a choice of about 100,000 ways (12!/7!) to discard, but it is only necessary to note that he now holds (a) fewer than four ♦'s or (b) fewer than four ♥'s. If (a), overtake ♠, ruff low ♦, ruff low ♥, ruff low ♦, ruff low ♥; declarer's hand is all good. If (b), do not overtake ♠, ruff low ♥, ruff low ♦, ruff low ♥, ruff low ♦; dummy is good.

July/Aug 1968

1 B		2 R	3 A	4 D	5 A	R		6 R		7 F		8 A		9 A		10 B
11 A	C		12 M	A	D		13 S	U	B	O	14 P	T	I	M	15 A	L
B		16 P		W		17 A		M		18 R	E	A		19 E	S	U
20 B	E	A	21 T		22 O	B	S	O	L	E	S	C	E	N	C	E
A		23 T	H	24 E	R	E		R		25 S	E	T		26 T	O	E
27 G	E	N	E	R	A	L	R	E	L	A	T	I	V	I	T	Y
E		28 A	N	G	L	E		D		29 W	A	C		30 A	S	E

Also solved by Winslow H. Hartford, '30, and Allen L. Zaklad, '65.

30 Above is the author's solution to the crossword puzzle.
Solutions were sent in by Avinash K. Dixit and Marion R. Hart, '13, who says it was "the first puzzle in *Technology Review* where I even understood the problem." And we're pleased to announce that a crossword or a double crosstick will be a regular monthly feature of the *Review* beginning in the fall.

31 Place some or all of the chess pieces belonging to one side on the board in such a way that none of them can legally move.

Here are three solutions from John L. Marshall, '67:

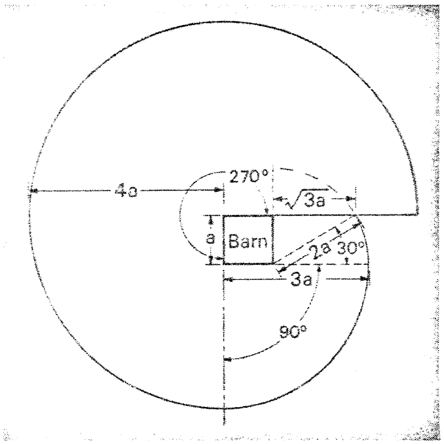
N	R	B	Q	K	B	R	N
	P	P	P	P	P	P	
	P					P	
	P					P	
	P		P	P	P	P	
N	R	B	R	K	B	R	N

B	R	B	R	K	B	R	B
	P		P	P	P	P	

Also solved by Donald R. Oestreicher, '67, Thomas D. Landale, S.M.'54, George Farnell, '41, and Allan Gottlieb, '67.

32 A cow is tethered to the corner of a square barn in a level field. The length of the tether equals the perimeter of the barn. The cow can graze over just *one acre.* What is the size of the barn, "give or take" a small decimal fraction?

The neatest solution is from Arthur L. Kaplan, '54:



Referring to the sketch, the one-acre (43,560 sq. ft.) grazing area of the cow is defined by the expression $3/4[\pi(4a)^2] + 1/4[\pi(3a)^2] + 1/12[\pi(2a)^2] + 1/2\,(\sqrt{3}a^2) = 4.3560 \times 10^4$ ft.^{2}
where a is the length of one side of the square barn. Solving this expression for a yields 30.547 = 30′ 6″ as the outside length of one side of the square barn.

Kenneth B. Blake, '13, adds:
I guess Frank G. Smith, '11, must have cut class the month you assigned problem 10. Problem 32 is the same, aside from the fact that 10 gave the size of the barn and asked for the area. Or does having the grazer a cow instead of a horse make a difference?

Also solved by Frank G. Smith, '11, John E. Prussing, '62, and Captain Roger A. Whitman, '61, whose answer came from "somewhere near Saigon."

33 Prove that

$$\sum_{n=1}^{k} n^2 = \frac{k(k+1)(2k+1)}{6}$$

Anthony D. Filardi, '46, says to use induction as follows:
For k = 1, the problem is trivial.
Assume the equation true for k = r.
In the case of k = r + 1:

$$\sum_{n=1}^{r+1} n^2 = \sum_{n=1}^{r} n^2 + (r+1)^2$$

$$= \frac{r(r+1)(2r+1)}{6} + (r+1)^2.$$

$$= (r+1)/6\,[r\,(2r+1) + 6\,(r+1)]$$

$$= (r+1)/6\,(2r^2 + 7r + 6)$$

$$= [(r+1)(r+2)(2r+3)]/6$$

Q.E.D.

Also solved by Thierry Labour, '72, John P. Rudy, '67 (the proposer, whose solution was hopelessly complicated and who obviously never heard of induction), Mr. Prussing, Mary A. Rogers (wife of Roddy R. Rogers, S.M. '57), Daniel S. Diamond, '65, Mary Lindenberg (wife of Martin S. Lindenberg, '39), Mr. Hovemeyer, and Captain Whitman.

34 Make 7 with two 2's.
The proposer, Smith D. Turner, '26, offers

$$\sqrt{\log \sqrt{\text{antilog}\,[\text{antilog}\,(2) - 2]}}$$

I offer:

$$\sqrt{\log \sqrt{\text{antilog}\,[\text{antilog}\,(2)]} - \log \sqrt{\text{antilog}\,(2)}}.$$

And Donald J. Cimilluca, S.M. '67, suggests: $d[(x \times x)/2 + x]/dx|_{x=2} = 7$.

Speed Department

SD11 For those who had trouble, a forfeit is recorded, 1-0.

Better Late Than Never

5 Mr. Hovemeyer has the following complaint about the problem numbered 5 in November:

I enjoy your column very much, but I do have one criticism: I feel that your attitude toward some of the problems as expressed in your column reflects what I feel is a typical engineer's view of mathematics. That is, I feel that your column suffers occasionally due to its not being precise as well as the fact that at times more concern is shown for displaying a solution than is shown for the method of solution. An example of what I am referring to is in problem 5 (November), which is to express the volume of a regular dodecahedron in terms of the length of an edge. The solution which you published was $V = 7.544\ a^3$, which is, of course, not true and is not even as good as the approximation of $V = 7.66312\ z^3$ given in the 14th edition of the CRC tables. If you are going to publish an approximation, why not at least use one which is close? Actually, I feel that neither of these solutions is satisfactory. Why not be precise? My solution to the problem is $V = 5/2\ a^3 \tan^2(3\pi/10) \tan [\sin^{-1}[1/(2 \sin \pi/5)]]$.

By the way, I am not an engineer—but rather a Ph.D. candidate in theoretical mathematics. The opinions (and styles) expressed by the responders in no way reflect those of the editor.

5 (December) A solution has come from Bill Dunbar.

7 Glenn A. Stoops, '61, Box 6442, Carmel, Calif., 93921, has some partial results and a closed-form solution for one part. He adds:

I don't care if I don't see this in print, but I had to get it off my chest. I do greatly enjoy your column, so much that I turn to it even before the '61 notes.

10 The answer printed was wrong, and correct estimates have now been submitted by Mr. Smith, John W. Callon, and Mr. Prussing.

15 In elementary number theory, the rationals [0, 1] have measure zero because if we order them like (0, 1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 1/n, 2/n, . . . (n — 1)/n, 1/(n + 1), . . .) we can cover them with open intervals of length e, (1/2)e, . . . $(1/2^n)e$, so measure of cover is $\leq 2e$. The problem is: Suppose $e = 1/10$; then the covering has length $\leq 1/5$. Exhibit a real number in [0, 1] which is not covered.

I am happy to say that John A. T. Munzer, '66, has earned a free subscription to *Tech Engineering News* for this first solution:

$$A = [(-1/20, 1/20) \cup (39/40, 41/40)] \cup$$

$$\bigcup_{n=2}^{\infty} \bigcup_{k=1}^{n-1} (k/n - \epsilon_{n,k},\ k/n + \epsilon_{n,k})$$

where $\epsilon_{n,k} = 2^{1-k}\epsilon_{n,1}$

$$\left.\begin{array}{l} \epsilon_{2,1} = 1/10\ 2^{-3} \\ \epsilon_{3,1} = 1/10\ 2^{-4} \\ \epsilon_{4,1} = 1/10\ 2^{-6} \end{array}\right\} \epsilon_{n,1} = 1/10 2^{-\frac{n^2-3n+8}{2}}$$

To demonstrate that $1/e \notin A$:

Suppose $|1/e - p/q| < \epsilon_{q,p}$ for $q \geqq 2$

then $|1/e - p/q| < \epsilon_{q,p} \leqq \epsilon_{q,1} = 1/10 \cdot 2^{-(q^2 - 3q + 8)/2} = 1/10 \cdot 2^{-[(q-3/2)^2]/2} \cdot 2^{-23/8} < 1/40 \cdot 2^{-[(q-3/2)^2]/2}$.

But $|1/e - p/q| = |1/e - [(q - 1)!\, p]/q! \geqq^* |1/e - (1/2! - 1/3! + \ldots \pm 1/p!)| = 1/[(q+1)!] - 1/[(q+2)!] + \ldots = X > 1/[2(q+1)!]$

*This inequality is true because

$X < 1/[(q+1)!] < 1/(2q!)$

This supposition has now led to:

$1/[2(q+1)!] < 1/40 \cdot 2^{-[(q-3/2)^2]/2}$

Let $f(q) = 20 \cdot 2^{-[(q-3/2)^2]/2}/(q+1)!$

and show $f(q) \geqq 1$ for $q \geqq 2$:

$[f(q+1)]/f(q) = \{(q+1)! \cdot 2^{[(q-1/2)^2]/2}\} / \{(q+2)! \cdot 2^{[(q-3/2)^2]/2}\}$

$= 2^{1/2\ [(q-1/2)^2 - (q-3/2)^2]}/(q+2 = 2^{(q-1)}/(q+2)$

$[f(q+1)]/f(q) < 1$ for $q = 2,3$

$[f(q+1)]/f(q) > 1$ for $q = 4$

So f(q) has a minimum at $q = 4$.

$f(4) = (20 \cdot 2^{25/8})/120 > 1$.

Note: $1/3 \notin (-1/20, 1/20) \cup (39/40, 41/40)$.

Other solutions have been received as indicated:

16 Eric Rosenthal (son of Meyer S. Rosenthal, '47).
17 John L. Joseph, '40.
20 E. W. Radtke and Mr. Yu.
24 Rolfe Petschek and Mr. Yu.

Allan J. Gottlieb, '67, is a graduate student in mathematics at Brandeis University. Address correspondence to him at the Department of Mathematics, Brandeis University, Waltham, Mass. 02154.

"Strobe Probe" Answers

1. The bullet shock wave has not reached the microphone; but a much weaker shock wave (barely visible in the picture) from the bullet's impact on the Plexiglas has triggered the microphone.

2. The ghostlike exposure beside the Pexiglas is reflection of light from the Plexiglas surface.