Puzzle Review

Allan J. Gottlieb, '67

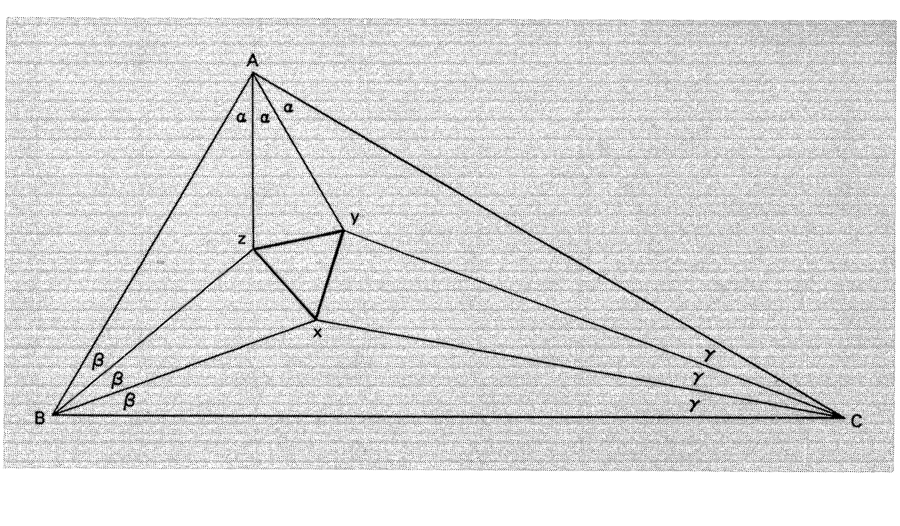
Last month while writing my column I was fighting a "life-and-death" battle with mononucleosis, which may explain the exceptionally low quality of that installment. A few weeks later I went home for Easter vacation. Of course, when my mother found out about "the disease," she was ready to hospitalize me—for up to six years, it seemed. Since this may be a problem encountered by others, I suggest that anyone involved look at *Reader's Digest* (a recent issue) for an article which puts mononucleosis in its place. Without this article, I would have been in bed my entire vacation and my mother wouldn't have slept a wink.

Next month (July) I will print no new problems. Instead, solutions to problems in both April and May issues will be given. This month's new problems will be answered in the first issue of *Technology Review* in the fall.

Problems

In problem 17 (Technology Review *for February)* you were asked to show that a certain triangle was equilateral. Now Eric Rosenthal, son of Meyer S. Rosenthal, '47, has a related question:

35 Given, as in the drawing below: We know from problem 17 that triangle XYZ is equilateral. Show that its sides are of length 8R sin α sin β sin γ, where R is the perimeter of triangle ABC.



The next two problems come from Douglas J. Hoylman, '64. The first one, he says, came out of an elementary calculus book, but four graduate students in mathematics required several hours to solve it. "Well, we solved it," he writes, "but our method used only algebra and trigonometry, and the problem appeared in a section on differentiation. Can anyone find a method of solution that uses calculus?" The problem:

36 A quadrilateral is inscribed in a circle such that one side is a diameter of the circle and the other three sides have lengths 1, 2 and 3, respectively. Find the length of the diameter to three decimal places.

Doug Hoylman's second problem:

37 Arrange six line segments of equal length in the plane to form eight equilateral triangles.

Russell A. Nahigian, '57, sent a long list of problems which will keep us going well into next year. One unusual one is:

38 Given four colored cubes described as follows:

	Front	Right	Back	Left	Top	Bottom
Cube 1	Green	Blue	Red	White	Red	Red
Cube 2	White	Green	Green	Red	White	Blue
Cube 3	Blue	Red	White	Blue	Green	Green
Cube 4	Red	White	Blue	Green	Red	White

The problem is to pile the blocks one above the other so that each face of the pile shows all four colors. (Cubes are helpful but not necessary to solve the problem.)

John Reed, '43, sends a puzzle which he believes has never been printed:

39 Find n such that $n^4 + n^3$ is a 10-digit number in which each digit is used only once.

Speed Department

From Mr. Nahigian:

SD13 Given VI = II. Move one stick to make the equality true. There are two solutions, and V ≠ II is not one of them!

Leon Sutton, '62, suggests:

SD14 John Dow usually arrives at the railroad station at 3 o'clock. His wife is always there to meet him and take him home (she always leaves the house at the same time and drives at a constant 30 m.p.h.) One day John's train arrives early, at 2:30, so he starts walking toward home. His wife, who left at the usual time, meets him on the way, picks him up, and they arrive home 10 minutes earlier than usual. When did his wife pick him up?

Solutions

20 If x and y are positive numbers with $x > y$, is x^y greater than, equal to, or less than y^x? Let $x > y \geq 0$, show that
a) if $y \geq e$, then $x^y < y^x$
b) if $x \leq e$, then $x^y > y^x$
c) if $y < 1$ and $e \leq x$, then $x^y > y^x$
d) if $1 < y < e$, then there exist infinitely many values of $x > e$ such that $x^y < y^x$ and infinitely many values of $x > e$ such that $x^y > y^x$ and exactly one value of $x > e$ such that $x^y = y^x$. And show that a corollary of this is the well-known fact that there is exactly one solution of $x^y = y^x$ for x and y integers, $0 < y < x$.

As I had suspected (feared), Professor Martin's problem was not too easy. After all, you don't become Chairman of the Mathematics Department by peddling grapefruit. Only two solutions were received. One, by William T. Moody, '31, included a geometric interpretation of $y^x = x^y$; the other, reprinted below, came from Mr. Hoylman:
Let $f(x,y) = y \log x - x \log y$. Then

$\partial f/\partial x = (y/x) - \log y$, and $\partial f/\partial y = \log x - x/y$.

a) If $y \geqq e$, then $\log y \geqq 1$, and $x > y$ gives $y/x < 1$, so $\partial f/\partial x < 0$. Hence for fixed y, f(x,y) is a decreasing function of x. Then, since $f(y,y) = 0$, we have $f(x,y) < 0$ for $x > y$, i.e. $y \log x < x \log y$, and since exp is an increasing function, $x^y < y^x$.

b) If $x \leqq e$, then $\log x \leqq 1$, and $x > y$ gives $x/y > 1$, so by a similar argument f (x,y) is a decreasing function of y, so $f(x,y) > 0$ for $y < x$, so $x^y > y^x$.

c) Let y be fixed. By elementary calculus (I quote Mr. Rosenthal's solution to problem 5), the maximum value of $x^{1/x}$ occurs for $x = e$. Hence $e^{1/e} > y^{1/y}$, or $e^y > y^e$. On the other hand,

$$\lim_{x\to\infty} (y \log x - x \log y) = \lim_{x\to\infty} x[y(\log x)/x - \log y] = -\infty,$$

so for sufficiently large x we have $y \log x < x \log y$, or $x^y < y^x$. So, by continuity, there is at least one x, $x > e$, such that $x^y = y^x$. Suppose there is more than one such x. Then there must be at least three (the curve starts above the x-axis and ends below it, so must cross it an odd number of times), or, equivalently, at least three solutions of $f(x,y) = 0$, where f is as above. This means there must be at least two solutions of $f_x(x,y) = 0$. But the latter equation has the unique solution $x = y/(\log y)$. Contradiction. To find a solution of $x^y = y^x$ in positive integers, $x > y$: by a) we must have $y < e$, i.e. $y = 1$ or 2. Clearly there is no solution with $y = 1$, so we must have $y = 2$. Then $x \geqq 3$, so $x > e$, and by c) the equation has exactly one solution in real numbers. But we know that it has the solution $x = 4$. Hence this is the only solution in integers.

21 The "Whitfield Six": Given

West	North	East
	♠ —	
	♥ 6 3	
	♦ A 9	
	♣ 8 2	
♠ 7 3		♠ 6 2
♥ —		♥ —
♦ K 10		♦ 8
♣ 9 5		♣ 7 4 3
	♠ 5 4	
	♥ —	
	♦ Q	
	♣ J 10 6	

South to lead, hearts trump; North-South to make all the tricks against any defense. The following comes from Peter J. Davis, Jr., whose father graduated in 1948:

Maxwell Smart might have called the solution the old end-game cross-dummy double-squeeze, but if you don't believe that try ruffing a spade in the dummy and leading the remaining trump. A club discard by East makes South's ♣ 6 good for the last trick, and a spade discard (with South dropping the ♦ Q) squeezes West into deciding whether to make the ♠ 5 good, to give the dummy two diamond tricks, or to give the last trick to the ♣ 8. This leaves East with his best and only remaining choice, to part with the ♦ 8, upon which South drops the ♦ Q and West the ♦ 7; any other discard by West provides South with a clearer path to success. South then cashes the dummy's ♦ A, and East has to make up his mind whether to give South a club trick or to allow his partner to decide which suit to lose the trick in.

The following observation is due to Ted E. Davis, '66:

It is noted that there are 10 clubs showing on the table, the missing cards being the ♣ A, ♣K, and ♣ Q. This situation could occur only if someone reneged when clubs were played or, if clubs were not played, the three clubs were sluffed on a heart lead (assuming there were no reneges).

And Lee H. Wilson, a University of Michigan journalism student, is puzzled:

What is impossible is figuring out who the clown, or clowns, were that sloughed away the ♣ A, ♣ K, and ♣ Q. My wife wouldn't even do that. Why not ask Mr. Zaklad to come up with an explanation? (*You just did.—Ed.*)

22 Let V be a closed convex set in a Hilbert space H. Let $x \in H - V$. Prove that there exists a unique $y \in V$ which is of minimal distance from x.

For a while I was afraid a physics major actually answered my challenge; in fact, I was worried: not only was his uniqueness proof identical to mine, but his existence proof was "better"! It would work for Banach spaces as well (i.e. he did not use the inner product). But after much scrutiny by Michael R. Gabel, '65, flaws were discovered. A result which Mr. Sutton simply asserted is in fact the heart of the existence proof. Here is his letter:

Let R be the suprenum of the radii of open balls about x which have a null intersection with V. (By open balls of radius r I mean, of course, the open set $\{z: \|x - z\| < r.\}$) Since V is closed (*and the space is complete and the parallelogram law holds for inner product spaces and etc.—Why don't you physics majors stick to naming new particles and leave Hilbert spaces to real men, i.e. mathematicians?—Ed.*), $\exists y \epsilon V$ s.t. $\|x - y\| = R$. Y is unique, for suppose $\exists\, y_1, y_2 \epsilon \|x - y_1\| = \|x - y_2\| = R$ ($y_1 \epsilon V$ and $y_2 \epsilon V$). Consider

$y' = y_1 + \lambda (y_2 - y_1)$
$= (1 - \lambda) y_1 + \lambda y_2 \ (0 \leqq \lambda \leqq 1)$.

Since V is convex, $y' \epsilon V$. (Y' lies along the line joining y_1 and y_2.) It is easy to show that

$\|x - y'\|^2 < R^2$ if $y_1 \neq y_2$:

$$\|x - y'\|^2 - \|x - y_1\|^2 = \lambda^2 \|y_1 - y_2\|^2 - 2\lambda \mathrm{Re}(x - y_1, y_2 - y_1). \quad (1)$$

In $\|y_1 - y_2\|^2$, substitute $y_2^2 = y_1^2 + 2\mathrm{Re}(x,y_2) - 2\mathrm{Re}(x,y_1)$, obtained from $\|x - y_1\|^2 = \|x - y_2\|^2 = R^2$. Expand both terms on the right-hand side of (1) to obtain: $0 \leqq \|y_1 - y_2\|^2$
$= 2y_1^2 + 2\mathrm{Re}(x,y_2) - 2\mathrm{Re}(x,y_1) - 2\mathrm{Re}(y_1,y_2)$
$= 2\mathrm{Re}(x - y_1, y_2 - y_1)$.

Therefore (1) becomes
$\|x - y'\|^2 - R^2 = (\lambda^2 - \lambda)\|y_1 - y_2\|^2$.

If $y_1 \neq y_2$, then, by choosing $0 < n < 1$, the right side is negative, so $\|x - y'\|^2 < R^2$, which contradicts the assumption that R = sup of the radii of the open balls which have null intersection with V. Therefore $y_1 = y_2$.

23 Given three co-planar circles with centers c_1, c_2, and c_3 radii equal to r_1, r_2, and r_3, respectively. Choose three points, one on each circle, and label them A, B and C. The problem is to find the maximum and minimum areas of such a triangle ABC, in terms of r_1, r_2 and r_3 and c_1c_2, c_2c_3 and c_3c_1.

Mark H. Yu, '70, the proposer, offered a year's subscription to *Tech Engineering News* for correct solutions, and it looks as if he was taking no chances. Russell L. Mallett, '57, shows the impossibility of a general construction as follows:

There is no general method of constructing the triangle of maximum or minimum area. By considering the case of three circles with the same radius r centered on the vertices of a right isosceles triangle with hypotenuse 4r, the existence of a ruler-and-compass solution becomes equivalent to the existence of rational solutions of the cubic equation $z^3 - z^2/8 - z/64 - 289/1024 = 0$. It is not difficult to show that this equation has no rational roots.

I do have a computer solution, however. (How about it, Mark?) Here is a letter from Francis T. Leahy, Jr., '33:

Mr. Yu's problem would be certainly difficult to solve using paper and pencil methods. It can, however, be solved easily with a Fortran program if this be considered a legitimate method. Points are chosen such that tangents to the circle through them are parallel to the opposite side of the triangle in question. To start off, the centers of the circle serve as the vertices of both triangles. In a very few iterations the vertices of the minimum and of the maximum triangle are obtained to any desired degree of accuracy, using an identical procedure except for sign in one equation. Here is a solution for a typical case:

```
      DIMENSION R(3),S(3),Q(3),A(4,2),THE(6)
      READ (5,10) R,S,Q
   10 FORMAT (10F4.0)
   12 FORMAT (1X,10F10.5)
      WRITE (6,13)
   13 FORMAT(91H1    X(1)      Y(1)    RAD(1)      X(2)      Y(2)    RAD(2)
     1X(3)      Y(3)    RAD(3)  )
      WRITE (6,12) (Q(J),S(J),R(J) ,J=1,3)
      DO 50 JA = 1,3
      DO 50 JB = 1,4,3
      JAC = JA + JB - 1
      A(JAC,1) = Q(JA)
   50 A(JAC,2) = S(JA)
      ARL =-1.
      DO121 JZ = 1,50
      DO221 JC = 1,4,3
      SN= 1.
      IF(JC.EQ.4) SN= -1.
      DO 20 JA = 1,3
      JAC = JA + JC - 1
      JB = MOD(JA,3) + JC
      D2 = A(JB,2) - A(JAC,2)
      D1 = A(JB,1) - A(JAC,1)
      THE(JA)=ATAN2(D2,D1)
      THE (JA + 3) = (D1**2 + D2 **2) **.5
   20 CONTINUE
      DO 21 JA = 1,3
      JAC = JA + JC - 1
      JB = MOD(JA    ,3) + 1
      A(JAC,1) = Q(JA)    - R(JA) * SIN(THE  (JB))* SN
      A(JAC,2) = S(JA)    + R(JA) * COS(THE  (JB))* SN
   21 CONTINUE
      SM = (THE(4)+THE(5)+THE(6))/2.
      AREA = (SM*(SM-THE(4))*(SM-THE(5))*(SM-THE(6)))** .5
      JA = JC
      JB = JC + 2
      WRITE (6,14)
   14 FORMAT (91H0    AREA   COOR. OF VERTEX(1)  COOR. OF VERTEX(2)  COO
     2R. OF VERTEX(3)                    )
      WRITE (6,12) AREA , ((A(J,K),K=1,2),J=JA,JB)
      IF(JC.EQ.1) GO TO 221
      IF(-ARL- ARK + AREA.LT..0001) STOP
      ARL = AREA - ARK
  221 ARK = AREA
  121 CONTINUE
      STOP
      END

    X(1)      Y(1)    RAD(1)      X(2)      Y(2)    RAD(2)      X(3)      Y(3)    RAD(3)
   0.        0.       1.00000   5.00000  11.00000   2.00000  11.00000   2.00000   3.00000

    AREA   COOR. OF VERTEX(1)  COOR. OF VERTEX(2)  COOR. OF VERTEX(3)
  55.50000   0.83205   0.55470   5.35777   9.03226   8.26990   3.24144

    AREA   COOR. OF VERTEX(1)  COOR. OF VERTEX(2)  COOR. OF VERTEX(3)
  55.50000  -0.83205  -0.55470   4.64223  12.96774  13.73110   0.75859

    AREA   COOR. OF VERTEX(1)  COOR. OF VERTEX(2)  COOR. OF VERTEX(3)
  35.44352   0.80346   0.44016   5.67056   9.11009   8.35381   3.41283

    AREA   COOR. OF VERTEX(1)  COOR. OF VERTEX(2)  COOR. OF VERTEX(3)
  94.86999  -0.80214  -0.59714   4.82037  12.99192  13.78078   0.87426

    AREA   COOR. OF VERTEX(1)  COOR. OF VERTEX(2)  COOR. OF VERTEX(3)
  35.24640   0.90591   0.42433   5.73841   9.14130   8.37362   3.44987

    AREA   COOR. OF VERTEX(1)  COOR. OF VERTEX(2)  COOR. OF VERTEX(3)
  94.94752  -0.80405  -0.59456   4.79922  12.98998  13.77209   0.85384

    AREA   COOR. OF VERTEX(1)  COOR. OF VERTEX(2)  COOR. OF VERTEX(3)
  35.23859   0.90745   0.42016   5.75007   9.14634   8.37637   3.45464

    AREA   COOR. OF VERTEX(1)  COOR. OF VERTEX(2)  COOR. OF VERTEX(3)
  94.94882  -0.80411  -0.59448   4.80235  12.99821  13.77334   0.85606

    AREA   COOR. OF VERTEX(1)  COOR. OF VERTEX(2)  COOR. OF VERTEX(3)
  35.23831   0.90806   0.41834   5.75282   9.14789   8.37697   3.45592

    AREA   COOR. OF VERTEX(1)  COOR. OF VERTEX(2)  COOR. OF VERTEX(3)
  94.94885  -0.80410  -0.59449   4.80197  12.99817  13.77312   0.85552

    AREA   COOR. OF VERTEX(1)  COOR. OF VERTEX(2)  COOR. OF VERTEX(3)
  35.23830   0.90811   0.41872   5.75336   9.14731   8.37698   3.45595

    AREA   COOR. OF VERTEX(1)  COOR. OF VERTEX(2)  COOR. OF VERTEX(3)
  94.94885  -0.80411  -0.59448   4.80203  12.99818  13.77314   0.85559
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24 Let N. be an R-digit number, written $n_0 n_1 \ldots n_{R-1}$, in the base R number system such that n_0 is the number of 0's in N, n_1 the number of 1's, and so on up to R—1. For what R does N exist? For which of these is it unique?

The only correct solution is from the proposer, Kenneth W. Dritz, '66:
I offer a derivation of two solutions for $R = 4$, one for $R = 5$ and a parametric one for each [R> 6]. The derivation is the proof that no other solutions exist.
Since the length of N is equal to the number of 0's in N, plus the number of 1's in N, and so on up to $R - 1$, the sum of the digits must be exactly R. Now let us assume we have a solution in which there are i non-zero digits ($n_0 = R - i$). Since no solution can have $n_0 = 0$, there are then $i - 1$ non-zero digits to the right of n_0. The sum of these $i - 1$ digits is $R - n_0 = i$. Therefore, one is a 2 and the other $i - 2$ are 1's. It is seen easily at this point that no solutions exist for $R = 2$ or $R = 3$. Ror $R > 3$ there are three cases:

Case 1: $n_1 = 0$. Since there are $i - 2$ 1's to the right of n_0 but none at all in N, $i - 2 = 0$ and $i = 2$. The 2 to the right of n_0 can only be n_2 (if $n_j = 2$ for some $j > 2$ there would have to be 2 j's in N in addition to the 2, contradicting $i = 2$). That takes care of all the non-zero digits but n_0. But since n_2 says we need two 2's in N, $n_0 = 2$. Since $2 = n_0 = R - i = R - 2$, $R = 4$ and one solution is 2020.
Case 2: $n_1 = 1$. Since there are $i - 2$ 1's to the right of n_0 and just one 1 in N (which happens to be right of n_0), $i - 2 = 1$ and $i = 3$. The 2 to the right of n_0 can only be n_2 (if $n_j = 2$ for some $j > 2$ there would have to be 2 j's in N in addition to the 2 and the 1, contradicting $i = 3$). Then $n_0 = 2$ for the same reason as before. Since $2 = n_0 = R - i = R - 3$, $R = 5$ and one solution is 21200.

Case 3: $n_1 = 2$. There are two subcases:
Case 3.1: Both of the 1's in N are to the right of n_0. Then $i - 2 = 2$ and $i = 4$. Since $n_1 = 2$, n_2 can't be 0. It can't be another 2 since there is only one 2 to the right of n_0. Hence $n_2 = 1$. Since n_0 must now be greater than 2, we have $2 < n_0 = R - i = R - 4$ and $R > 6$. Obviously, the remaining 1 has to be n_{R-4} (which is guaranteed to be to the right of n_2). The parametric solution for $R > 6$ is, then, $(R - 4)\,21\underbrace{0\ldots0}_{R-7\ 0\text{'s}}1000$.
Case 3.2: Only one of the 1's in N is to the right of n_0. Then $n_0 - 1$; $i - 2 = 1$ and $i = 3$. Since $n_1 = 2$, n_2 can't be 0. Nor another 2 (as before). Hence, n_2 is the remaining 1. Since $1 = n_0 = R - i = R - 3$, $R = 4$ and one solution is 1210.
We have exhausted all the possibilities. So there is no solution for $R = 6$. John's solution, 6210001000, for $R = 10$, falls out of Case 3.1.

Messrs. Hoylman, Sutton, and Yu sent in identical answers (they each missed 1210). Do you three know each other?

Better Late Than Never

5 Lawrence H. Bailey, '15, has responded to this problem.

7 R. Robinson Rowe, '18, has extended further the solution of this problem, calling attention to his earlier solution of part (a):
My solution was $y = e^{Dx}$, where $D = 0.567143290409784\ldots$ Testing this,
$y' = De^{Dx} = Dy$
$f(x - 1) = e^{D(x-1)} = e^{Dx}e^{-D} = e^{-D}y$
If $y' = f(x - 1)$, $Dy = e^{-D}y$
and $D = e^{-D}$,
which determines D as given above.

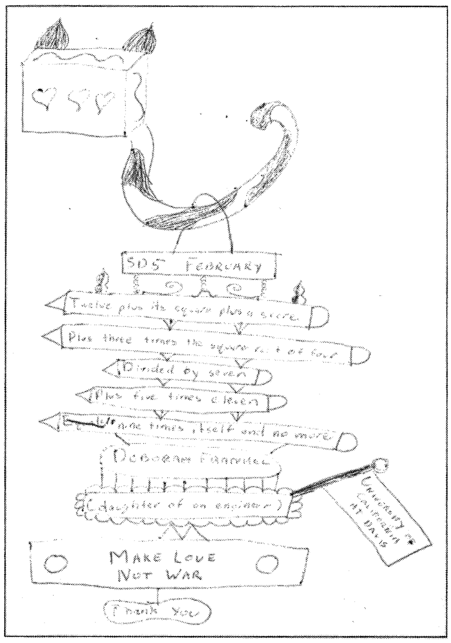
There is also an extended discussion from Reino W. Hakala, Professor and Chairman of the Department of Mathematics at Oklahoma City University, who adds a postscript: If you know of any recent Ph.D.'s in mathematics or anyone who will receive a Ph.D. in mathematics before next September who would like to teach at Oklahoma City University (the former Miss America is a student here, and there are other advantages as well), I would appreciate knowing of them.
I am looking for someone of that sort to add strength to our small department.
I prefer someone with a background in topology and/or functional analysis but will consider any background in pure or applied mathematics.

8 Mr. Sutton has responded.

10 Milton S. Hathaway, '32, has a rather interesting observation:
Don't see how you and all those other fellows can be wrong about that horse and that barn. A full grazing circle on that 100′ rope would be 31,416 sq. ft. But it seems to me that 25′ square barn would make poor grazing, so the real grazing area cannot be more than 31,416 — 625 = 30,791 sq. ft., so I doubt your answer of 30,920.
How about it, Marshall Greenspan, '61?

12 Arnold B. Staubach, '19, has replied.

SD5 This has got to be the most aesthetic answer ever received by Puzzle Review:



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