Puzzle Review

Allan J. Gottlieb, '67

Lately several readers have been requesting that I phone them to discuss some of the problems. I am afraid that this is impossible on my schedule. As I have said before, that would destroy the essence of mass communication. Please submit letters and state your grievances in writing to me at the Mathematics Department, Brandeis University, Waltham, Mass., 02154. Since often many people notice the same mistakes or ambiguities (there always seem to be some), one correction will suffice.

I have been asked what I plan to do to avoid the draft next year. Should the present situation remain, I most likely will enlist. Perhaps next year my address will include A.P.O. Does the Army need puzzle editors?

Problems

The first problem was sent in by Mark D. Horowitz, '71, about three months ago. I must apologize for the unreasonable delay in publication, but we thought we may have printed it before and ensuing investigations accounted for most of the delay. Here it is at last:

25 Given a rectangular solid box with inside dimensions of 18"x51"x69" and incompressible golf balls 1.82" in diameter, find the maximum number of golf balls that can be packed into the box. Assume that gravity is present and that the box may be rotated to any spacial orientation, so long as it is not deformed.

The following arrived from Bojan Popović, a mathematics student in Belgrade, Yugoslavia; I was very pleasantly surprised to see that my column has traveled so far:

26 On the table are given six matches of equal length. Make four identical triangles without breaking them.

The next two problems are from Shih-Ping Wang, S.M.'61, and they have also been somewhat delayed in their appearance:

27 Find the solution for the case of two odd balls both defective by the same absolute amount (but they may be opposite in sign).

28 Along the same vein, using a two-pan balance: given a set of n weights, each of integral weight $w_1, w_2, \ldots w_n$ such that any object with an integral weight from unity to $w_1 + w_2 + \ldots w_n$, the sum of all weights can be determined.

This problem arrived from Winslow H. Hartford, '30, along with these comments: "Just a line to let you know how much I enjoy Puzzle Review. You are running Martin Gardner of *Scientific American* a good second; I've met a lot of old favorites, like the coconut problem. I don't have any football problems handy, but I suppose the type of problem which ends up 'the engineer's name is Smith' could readily be done over to end up 'the flanker back's name is Wojchieszczojyski.' Here is an old favorite of mine—a bridge problem which is a stinker."

29 With the following, South is the declarer at seven spades. West leads ♣8. The problem is to make seven against any defense.

	♠ K J 9 7 6 ♥ A J 9 6 5 ♦ — ♣ A J 9	
♠ Q 10 8 ♥ K ♦ K 3 2 ♣ 8 7 6 5 4 3		♠ — ♥ Q 10 8 7 4 3 2 ♦ Q 10 8 7 4 ♣ K
	♠ A 5 4 3 2 ♥ — ♦ A J 9 6 5 ♣ Q 10 2	

Speed Department

SD9 Joseph W. Lovell, '13, asks you to find the fallacy in his proof: given that $a = b + c$, then $a = b$. This is the proof:

$a = b + c$

$a(a - b) = (a - b)(b + c)$

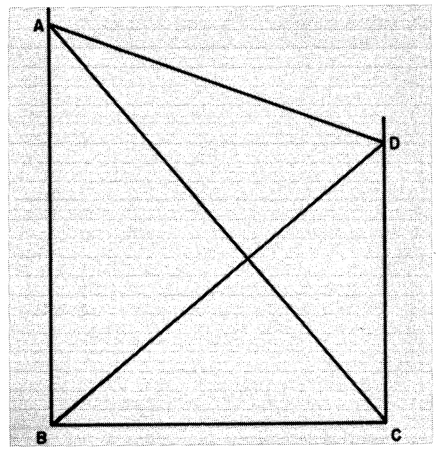
$a^2 - ab = ab + ac - b^2 - bc$

$a^2 - ab - ac = ab - b^2 - bc$. Then

$a(a - b - c) = b(a - b - c)$, or

$a = b$

SD10 Richard P. Bishop, '59, submitted the following problem of Dr. Murray Spiegel:



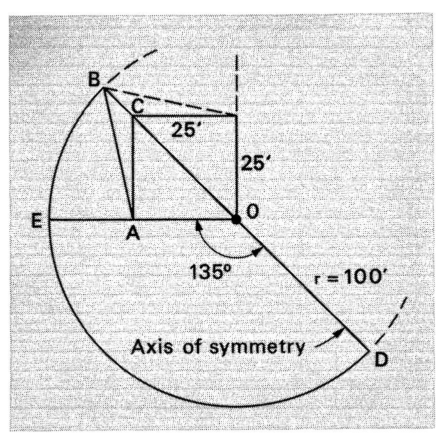
Given that angle ABC = angle BCD = 90°, angle DBC = 40° and angle BCA = 50°, find angle CAD using only a straight edge and compass.

If the solutions to Speed Department problems are difficult, I will print them.

Solutions

10 A farm horse is tethered to one corner of a barn 25 feet square, in the middle of an open field, with a rope 100 feet long. What is the area the horse can graze on?

The following solution is by Marshall Greenspan, '61:
In the following drawing, the area for grazing equals twice the area of sector EOD plus twice the area of sector BAE plus twice the area of triangle ABC.



The first step is to determine the angle BAC, which equals $\sin^{-1}(25/75 \sin 135°)$, which is 13°38′. Therefore angle BAC equals 180° — 135° — 13°38′, or 31°22′. Next, determine angle BAE which equals 90° — angle BAC, or 58°38′. Now we can determine the areas:
BAC = ½(25) (75 sin 31°22′) = 800 sq. ft.
BAE = $\pi(75)^2(58°38'/360°)$ = 2880 sq. ft.
EOD = $\pi(100)^2(135°/360°)$ = 11780 sq. ft.
The total of these areas is 15460 sq. ft.; the total grazing area is therefore 30920 sq. ft.
To check this result, note that $\pi(100)^2 = 31400$, which is greater than 30920.

Also solved by Douglas K. Severn, '23, Eric Rosenthal (son of Meyer S. Rosenthal, '47), Mark H. Yu, '70, Douglas J. Hoylman, '64, Kenneth B. Blake, '13 (who adds that "my old classmate Howard S. Currier, '13, will have to dig up a tougher problem if he wants to stop me!"), Arnold B. Staubach, '19, and Mark and John D. Pfeil (sons of John S. Pfeil, Jr., '43).

11 How is it possible for a batter to get a hit and thus raise his batting average exactly one point? (A trivial solution may immediately come to mind, namely, the batter who has gone hitless for his first 999 times at bat and then gets a hit to raise his average from .000 to .001. Nontrivial solutions are desired.)

Richard Haberman, '67, sent in this solution:
Let x = number of hits and y = number of times at bat. The problem is to solve for x and y, positive integers, $x \leq y$, such that $(x + 1)/(y + 1) - x/y = .001$.
Consequently, $x = .001\, y(999 - y)$; for x to be an integer, $y(999 - y) = 1000n$, where n is a positive integer. Therefore $y^2 - 999y + n1000 = 0$.
The quadratic formula implies
$y = [999 \pm \sqrt{(999)^2 - 4(1000)n}]/z$.
For y to be an integer, $(999)^2 - 4(1000)n$ must be a perfect square $\equiv z^2$; but $(999)^2 = 998001$ and hence $\mathrm{mod}_{1000}(z^2) = 1$.
Furthermore $z \leq 999$. A quick look through C.R.C. shows the only numbers (0, 999) which end 001 are 1, 249, 251, 499, 501, 749, 751, and 999. We know that
$n = ¼\ [(999)^2 - z^2]/1000$
and we easily see that z = 249 is the *only* solution except the trivial one (z = 999):

z	z^2	$[(999)^2 - z^2]/1000$	n
1	1	998	not integer
249	62001	936	234
251	63001	935	not integer
499	249001	749	not integer
501	251001	747	not integer
749	561001	437	not integer
751	564001	434	not integer
999	998001	0	0

Hence $y = (999 \pm 249)/2 = 624$ or 375, and x = 234. The two possibilities are then:
234/624 = ⅜ = .375 and 235/625 = 47/125 = .376;
234/375 = 78/125 = .624 and 235/376 = ⅝ = .625.
(Notice the symmetry.) Both are great hitters!
The "big three" solved this as well—namely, Messrs. Yu, Rosenthal and Hoylman—and so did Mr. Greenspan and Arthur W. Anderson, '63, who included a small epistle for his proof.

12 It should always be possible to solve this equation
$x^2 + 2xy + y = 4uv + u - v$
in positive whole numbers (greater than 0) with arbitrary values assigned to either x and y or u and v.

Arnold B. Staubach, '19, writes as follows: "The solution may consist of the application of the New Math ideas of arrays and one-to-one correspondence, which reveal the hidden structure of the expressions on each side of the equation, together with proof by induction, as shown in the following diagrams and equations."

y \ x	1	2	3	4	
1	4 ①	9 ②	16 ③	25 ④	···
2	7	14	23	34	···
3	10	19	30	43	···
4	13	24	37	52	···
	⋮	⋮	⋮	⋮	

$x^2 + 2xy + y$
Vertical increment: 3 5 7 9
Value of y = 1: $(x + 1)^2$
Increment of y: $(2x + 1)(y - 1)$
Sum: $(x^2 + 2x + 1) = (2xy + y - 2x - 1)$
Reduces to: $x^2 + 2xy + y$

v \ u	1	2	3	4	
1	4 ①	9 ②	14	19	···
2	7	16 ③	25 ④	34	···
3	10	23	36	49	···
4	13	30	47	64	···
	⋮	⋮	⋮	⋮	

$4uv + u - v$
Vertical increment: 3 7 11 15
Value of v = 1: $(5u - 1)$
Increment of v: $(4u - 1)(v - 1)$
Sum: $(5u - 1) + (4uv - 4u - v - 1)$
Reduces to: $4uv + u - 1$

Unfortunately, I can't understand this solution but it looks so interesting that I am printing it anyway. If anyone can understand it, especially Mr. Staubach, I would appreciate a letter. Messrs. Hoylman and Yu submitted partial solutions.

13 A community with N institutions of learning decides to form a football league. Each team was to play every other team once, each team was to have one idle weekend during the season, and no team would play two consecutive games either at home or away. Only one team could be idle on a given weekend. What are the chances of delivering a schedule?

The following super-elegant solution is by James E. Ruttenberg, '63 (my money says he's a math major):
Let N = 2M + 1 be the number of teams. We shall construct an N × N matrix representing the schedule. The ith row shall represent the schedule for the ith week, to wit: a_{i1} designates the team that is idle on the ith week. The remaining 2M elements are considered as M ordered pairs, the first of a pair being the home team, who are scheduled with the second of the pair, the away team. The requirements are:
1. Each row shall contain the integers 1, . . , N exclusively; i.e., each week, each team either plays or is idle.
2. The first column contains the integers 1, . . , N exclusively; i.e., each team is idle exactly once.
3. If a team (integer) appears in an odd column one week, it must appear in an even column the next week, and vice versa; i.e., a team alternates home and away. Column 1 is ignored for this consideration.
The following matrix satisfies these requirements:

$$a_{ij} = \begin{cases} i & \text{if } j = 1 \quad i = 1, N \\ i + 1 & \text{if } j = N \quad i = 1, N \\ i + j & \text{if } 2 \leq j \leq N - 1 \quad i = 1, N \end{cases}$$

Addition is taken modulo N, and we take N mod N = N, not N mod N = ϕ.
Requirements 1. and 2. are obviously satisfied. As for 3., the appearance of the integer k in the ith row implies i + j = k for some j. In the next row, i has increased by 1, so that j must decrease by 1 for k to remain constant. An increase of 1 means a change in parity. Transitions into and out of the first and last columns are seen to satisfy this parity shift.
The following is an example for N = 5:

1	3	4	5	2
2	4	5	1	3
3	5	1	2	4
4	1	2	3	5
5	2	3	4	1

Obviously, any mapping that is 1-1 from the set (1, 2, . . , N) to itself will preserve the schedule. Hence there are N! solutions, each one reflecting a different permutation of the order of teams idle.

Also solved by John E. Prussing, '62, Leo P. Buckley, Jr., '52, and Messrs. Yu and Hoylman.

14 Prove that a nonstandard ball can be determined in n weighings from a set of $(3^n - 1)/2$ plus 1 balls, one of which is marked as standard.

Here is the solution of the proposer, Charles D. Coltharp, '58:
Let Q_n be the proposition that a nonstandard ball can be determined in n weighings from a set of
$(3^n - 1)/2$ pl + $(3^n + 1)/2$ ph or
$(3^n - 1)/2$ ph + $(3^n + 1)/2$ pl, where
pl = possibly light ball or balls, and
ph = possibly heavy ball or balls.
Q_1 is true, since given one pl and two ph, one ph can be weighed against the other and an imbalance will indicate which is heavy. A balance will indicate that the pl is light. Because of the symmetry in the problem, pl and ph can be reversed, so the case of two pl and one ph does not need to be considered separately. In what follows, appeals to symmetry will be implied rather than explicit. P_2 is true since two unknown balls can be weighed against an unknown ball and the marked ball, and a balance will indicate that the unweighed ball is nonstandard. It can be weighed against the marked ball to determine if it is heavy or light. An imbalance reduces the problem to Q_1. Now suppose P_n and Q_{n-1} to be true, and examine P_{n+1} and Q_n. Set $(3^n - 1)/2$ balls aside and 3^n balls remain, since
$(3^{n+1} - 1)/2 - (3^n - 1)/2 = (3^{n+1} - 3^n)/2 = 3^n(3 - 1)/2 = 3^n$.
Weigh $(3^n - 1)/2$ + the marked ball against $(3^n + 1)/2$ balls. If a balance occurs, the problem reduces to P_n, which is true. If an imbalance occurs, we have, without loss of generality, $(3^{n-1})/2$ pl and $(3^n + 1)/2$ ph, which is Q_n. Set aside $(3^{n-1} - 1)/2$ pl and $(3^{n-1} + 1)/2$ ph, and there remains 3^{n-1} pl plus 3^{n-1} ph. Weigh $(3^{n-1} + 1)/2$ pl + $(3^n - 1)/2$ ph against $(3^{n-1} - 1)/2$ pl + $(3^{n-1} + 1)/2$ ph, and no matter what happens the problem reduces to Q_{n-1}, which is true. Therefore P_n and Q_{n-1} imply P_{n+1} and Q_n. Couple this with the truth of P_2 and Q_1, and we have by induction that P_n is true for all n.

Also solved by Mr. Yu.

SD4 Let n_0 be a number, n_1 be the number of letters in the spelling (in English) of the number $n_0, \ldots, n_k$ be the number of letters in the spelling of n_{k-1}. Prove $\lim_{k \to \infty} n_k = 4$, and show that this is independent of the language used.

In general I do not print solutions to speed problems, but for this one I will make an exception. Mr. Hoylman's is the most interesting:
It is easily seen that for $n > 4$, the number of letters in the name of n is less than n. (If you don't believe me, write them all out.) Hence the sequence $n_0, n_1, n_2, \ldots$ is strictly decreasing until one of the n_k is less than 5. Then we have

$$\left.\begin{array}{l}\text{one}\\ \text{two}\end{array}\right\} \rightarrow \text{three} \rightarrow \text{five} \rightarrow \text{four.}$$

So eventually 4 must appear in the sequence. But if n_k is 4, so is n_{k+1}, and all terms after that. So the sequence converges to 4. I can't figure out what he means by, "Show this is independent of the language used." Indeed, in German the same thing occurs, and you eventually get stuck on "vier," but in Spanish you could either stick with "cinco" or oscillate infinitely between "cuatro" and "seis," and in French you keep going around the circle:

$$\begin{array}{ccc}\text{quatre} & \rightarrow & \text{six}\\ \uparrow & & \downarrow\\ \text{cinq} & \leftarrow & \text{trois}\end{array}$$

Furthermore, in Old High Martian, the name for the number N has N + 1 letters, so the sequence would tend to infinity.

Also solved by Donald E. Savage, '54, and Mr. Rosenthal.

Better Late Than Never

Solutions to the following problems have come from those indicated:

79 Richard P. Bishop, '59, and Eric Hovemeyer.
3 John F. Simmons.
5 R. Robinson Row, '18, and Mr. Hovemeyer.
7 Charles S. Sutton, '35.
8 Jeffrey D. Dodson, '67, and Mr. Sutton.

Allan J. Gottlieb, '67, is a graduate student in mathematics at Brandeis University. "Puzzle Review" is written for *Technology Review* and *Tech Engineering News,* the M.I.T. undergraduate professional magazine.