Puzzle Review

Allan J. Gottlieb, '67

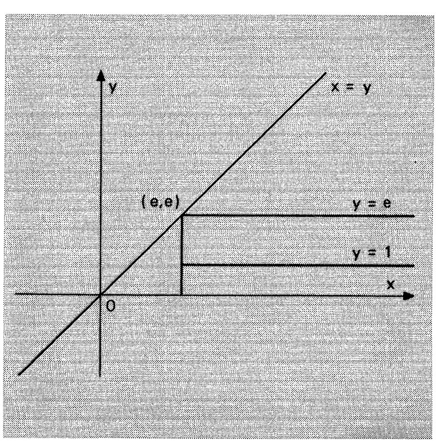
I have a personal problem whose solution I would really welcome. Why is it that no matter how long I spend studying for an exam, it always turns out that I needed about two hours more? Anyone who explains this to me and includes a solution which works will receive six free issues of both *Technology Review and Tech Engineering News,* even if I have to pay for them myself.

A very conscientious reader phoned me a few weeks ago to ask where he should send his solutions. I must apologize for not including that information in each installment. This oversight shall be corrected: address all correspondence to me at the Department of Mathematics, Brandeis University, Waltham, Mass., 02154.

Problems

20 This month's first problem was submitted by the Head of the M.I.T. Mathematics Department, William Ted Martin:

In problem 5 (*see* Technology Review *Dec., 1967, p. 62*) it is asked whether π^e is greater than, equal to, or less than e^π. While this may be answered by computation or by other means, it is a special case of the following question: If x and y are positive numbers with $x > y$, is x^y greater than, equal to, or less than y^x? The readers are asked to prove the following:



Let $x > y \geq 0$ show that
a) if $y \geq e$, then $x^y < y^x$
b) if $x \leq e$, then $x^y > y^x$
c) if $y < 1$ and $e \leq x$, then $x^y > y^x$, and
d) if $1 < y < e$, then there exist (infinitely many) values of $x > e$ such that $x^y < y^x$ and (infinitely many) values of $x > e$ such that $x^y > y^x$, and exactly one value of $x > e$ such that $x^y = y^x$. (Show that a corollary of this is the well-known fact that there is exactly one solution of $x^y = y^x$ for x and y integers, $0 < y < x$.)

21 The following came from Allen L. Zaklad, '65:
To introduce myself, I graduated from the 'Tute in '65 in mathematics and am now at the University of Pennsylvania in psychology. Being an old bridge player, I've decided to play with your problems and give you a couple of my own. One of them:

This is an end-game double-dummy problem, one of the classics of the game, called the "Whitfield Six":

North: ♠ — ♥ 6 3 ♦ A 9 ♣ 8 2

West: ♠ 7 3 ♥ — ♦ K 10 ♣ 9 5

East: ♠ 6 2 ♥ — ♦ 8 ♣ 7 4 3

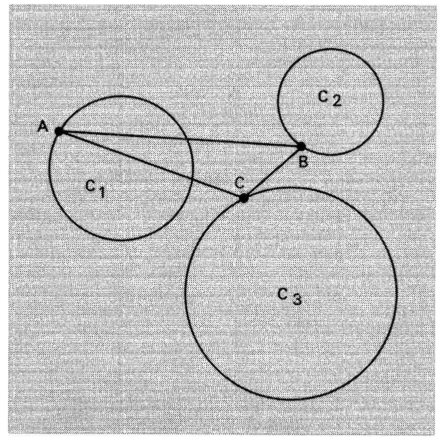
South: ♠ 5 4 ♥ — ♦ Q ♣ J 10 6

South to lead, hearts trump; North-South to make all the tricks against any defense.

22 My girl friend, a physics major at Brandeis, has told me that all contemporary physicists have the Hilbert space theory "scoped out" (understood completely). Here is a very elementary problem about Hilbert spaces to test her hypothesis:

Let V be a closed convex set in a Hilbert space H. Let $x \in H - V$. Prove that there exists a unique $y \in V$ which is of minimal distance from x.

23 Here is a letter from Mark H. Yu, '70, "Senior House's answer to Martin Gardner":
Here's an original problem that I thought might be of interest to you:



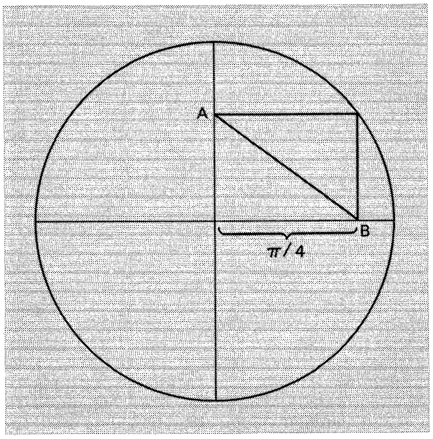
Suppose we are given three co-planar circles with centers c_1, c_2, and c_3 and radii equal to r_1, r_2 and r_3, respectively. Choose three points, one on each circle, and label them A, B and C. The problem is to find the maximum and minimum areas of such a triangle ABC, in terms of r_1, r_2 and r_3 and $\overline{c_1c_2}$, $\overline{c_2c_3}$ and $\overline{c_3c_1}$. (Admittedly, this is a hard problem. I solved it for the maximum-area case through analytic geometry and various transformations. As for the minimum-area case, I derived four fourth-degree equations (or something like that). I believe that these problems *can* be solved using transformations only (i.e., $\triangle ABC$ can be constructed legally) by someone who is more competent than I. I'm willing to give a year's subscription to *Tech Engineering News* for correct solutions.

24 The last regular problem is from Kenneth W. Dritz, '66, who writes:
Your *Technology Review* readers might be interested in the following generalization of John H. Boynton's ('58) problem 1 (*see* Technology Review, *Oct./Nov., 1967, p. 74):*

Let N be an R-digit number, written $n_0n_1 \ldots n_{R-1}$, in the base R number system, such that n_0 is the number of 0's in N, n_1 the number of 1's, and so on up to $R - 1$. For what R does N exist? For which of these is it unique?

Speed Department

SD7 George A. W. Boehm asks the length of the line AB on the following page:



Old Tech Engineering News *readers may find this one familiar.—Ed.*

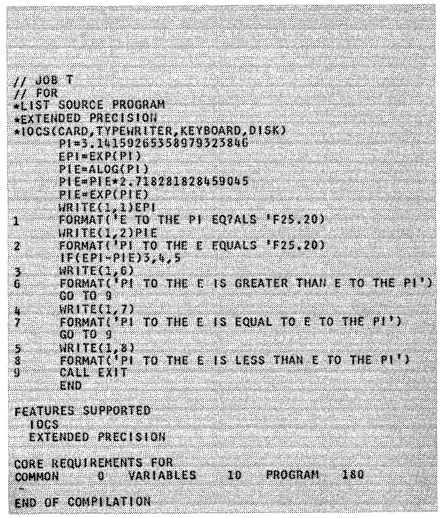
SD8 Make 100 with four 7's (i.e., 7/7 — 7/7 or $(7 + 7)^{7/7}$, etc.).

Solutions

5 Is π^e greater than, equal to, or less than e^π?

A theoretical solution came from Eric Rosenthal (son of Meyer S. Rosenthal, '47):
You can use elementary calculus to find $(e, e^{1/e})$ is a maximum point of $y = x^{1/x}$ and also is the only maximum or minimum.
So $e^{1/e} > \pi^{1/\pi}$
$(e^{1/e})^{e\pi} > (\pi^{1/\pi})^{e\pi}$
$e^\pi > \pi^e$.

Readers may also be interested in the following from Robert Fitch (son of John T. Fitch, '52):
I am a junior at Concord-Carlisle (Mass.) High School and I have obtained an answer to problem 5 using an I.B.M. 1130 computer. I have enclosed a print-out of the program I wrote (above, right). It is written in FORTRAN and works by computing values for e^π and π^e. It then compares the two and prints the result. The execution of the program occurs at the bottom of the print-out, with the values of the two numbers being typed out and then the result of the comparison. As you can see, π^e is less than e^π.



```
// JOB T
// FOR
*LIST SOURCE PROGRAM
*EXTENDED PRECISION
*IOCS(CARD,TYPEWRITER,KEYBOARD,DISK)
      PI=3.14159265358979323846
      EPI=EXP(PI)
      PIE=ALOG(PI)
      PIE=PIE*2.718281828459045
      PIE=EXP(PIE)
      WRITE(1,1)EPI
1     FORMAT('E TO THE PI EQ?ALS 'F25.20)
      WRITE(1,2)PIE
2     FORMAT('PI TO THE E EQUALS 'F25.20)
      IF(EPI-PIE)3,4,5
3     WRITE(1,6)
6     FORMAT('PI TO THE E IS GREATER THAN E TO THE PI')
      GO TO 9
4     WRITE(1,7)
7     FORMAT('PI TO THE E IS EQUAL TO E TO THE PI')
      GO TO 9
5     WRITE(1,8)
8     FORMAT('PI TO THE E IS LESS THAN E TO THE PI')
9     CALL EXIT
      END

FEATURES SUPPORTED
  IOCS
  EXTENDED PRECISION

CORE REQUIREMENTS FOR
COMMON      0   VARIABLES    10   PROGRAM   180

END OF COMPILATION
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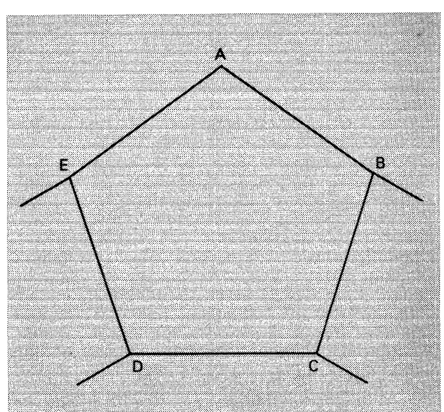
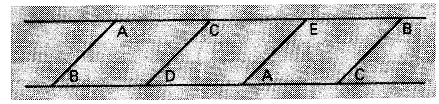
Also solved by Richard J. Grant, '65, John P. Rudy, '67, Douglas J. Hoylman, '64, Beverly Seavey, James P. Friend, '51, Howard W. Nicholson, Jr., '66, Andrew D. Egendorf, '67, and Jan M. Chaiken, Ph.D. '66.

6 Take a strip of paper with parallel edges; tie an overhand knot, making sure there is no looseness where the free ends "leave" the knot. Hold it up to the light source and observe all but one diagonal of a five-point star. Do the five points determine a *regular* pentagon?

Mr. Rosenthal sent in a model to substantiate his claim. Mr. Grant submitted the following sketch of a proof:
A cute way to do this is actually to fold the thing (including making creases at DE and BC) and write down all the obvious equalities from symmetry:



AE = AB, ED = BC, angle E = angle B, angle D = angle C.
Then label each corner *on every layer.*
Then unfold. You get something like this:



(I made up the letters because my wife threw out my model when I was playing with it; she thought I should be studying for orals.) Anyway, apply "alternate interior angles, parallel lines" as much as possible, and that's it.

7 The solution of this differential equation is familiar:
$dy/dx = y$, $y(0) = 1$.
But what is the solution if one makes a slight change in the differential equation:
$dy(x) / dx = y(x - 1)$, $y(0) = 1$?
Using this idea of writing equations, two other problems are suggested:
$dy/dx = y(x^2)$, $y(0) = 1$, and
$dy/dx = y[y(x)]$, $y(0) = 1$.
Mr. Hoylman appears to have done the best job:
The natural thing to try would seem to be a series solution. This works beautifully for one problem, runs into a snag on the second, and looks incredibly hairy for the third. First the beautiful one.
Given the problem:
$f'(x) = f(x^2)$, $f(0) = 1$,
we try to find a solution which is analytic in a neighborhood of 0, hence can be written as a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Then the equation becomes

$$\sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_n x^{2n}$$

and $a_0 = 1$, or
$a_1 + 2a_2x + 3a_3x^2 + \ldots =$
$1 + a_1x^2 + a_2x^4 + a_3x^6 + \ldots$

Setting the corresponding coefficients equal, we get
$a_1 = 1$, $2a_2 = 0$, $3a_3 = a_1$, $4a_4 = 0$, $5a_5 = a_2, \ldots$, or
$a_{2n} = 0$, $a_{2n+1} = a_n/(2n + 1)$. Then the only non-zero coefficients are 1, 2 + 1, 2(2 + 1) + 1, . . . , which are just the numbers of the form $2^m - 1$. Then we have

$$a_n = \prod_{k=1}^{m} (2^k - 1)^{-1} \text{ if } n = 2^m - 1,\ a_n = 0$$
otherwise. (Someone else may be able to express that product in closed form; I couldn't.) Hence the *unique* analytic (at 0) solution to the problem is:

$$f(x) = 1 + x + x^3/3 + x^7/21 + x^{15}/315 + \ldots$$

which converges for $1 x 1 \leq 1$. We have

$$f'(x) = f(x^2) = 1 + x^2 + x^6/3 + x^{14}/21 + \ldots$$

Now try the same thing with the equation $f'(x) = f(x - 1)$, $f(0) = 1$. (Incidentally, $\cos x$ is a particular solution to the similar problem $f'(x) = -f(x - \pi/2)$, but I couldn't see how to get a solution to this problem out of it.) To apply the power series method, we must make the very strong assumption that the radius of the convergence of the solution is greater than 1, so that the expansion

$$f(x - 1) = \sum_{n=0}^{\infty} a_n(x - 1)^n$$

makes sense. We have

$$f(x - 1) = \sum_{n=0}^{\infty} a_n[x^n - nx^{n-1} + \ldots + (-1)^n(n/2)x^2 - (-1)^n nx + (-1)^n]$$

$$= \sum_{n=0}^{\infty} (-1)^n a_n + \sum_{n=1}^{\infty} (-1)^n n a_n x + \ldots + \sum_{n=m}^{\infty} (-1)^n (n/m) a_n x^m$$

$$= \sum_{m=0}^{\infty} \left[\sum_{n=m}^{\infty} (-1)^n (n/m) a_n \right] x^m$$

where $n/m = n!/[m!(n - m)!]$

(The rearrangements make sense because the series is absolutely convergent in a neighborhood of 0.) Comparing this with the series for $f'(x)$, we get

$$(m + 1)a_{m+1} = \sum_{n=m}^{\infty} (-1)^n (n/m) a_n$$

very neatly specifying each coefficient in terms of the infinitely many that come after it. Now how do you solve something like that? (Well, you could if the series terminated, but it's easy to see that no polynomial solves the equation.) As for the third equation, I'm willing to let somebody else try to do something with

$$\sum_{n=0}^{\infty} a_n \left[\sum_{m=0}^{\infty} a_m x^m \right]^n.$$

There was an even more involved discussion of this problem from R. Robinson Rowe, '18, who writes that he conducted a puzzle column in *Civil Engineering* for 19 years (thanks for the good time and yes I do enjoy the column), and there was also a solution from Mr. Grant.

8 The first, second, and third derivatives of functions are familiar, but what is a ½ derivative, a πth derivative, or an ith derivative?

A former colleague of mine, Daniel A. Asimov, '68, has a fine solution:

Fractional derivatives can be defined in several (unfortunately unequal) ways, all satisfying the natural requirement $D^a(D^b f) = D^{a+b} f$. One of my favorite ways is as follows: Notice that $D^n(x^k) = k(k - 1) \ldots (k - m + 1)x^{k-n}$. Now recall the generalization of factorial to real numbers, the gamma function Γ satisfying $\Gamma(m) = (m - 1)!$ for m a positive integer. Since

$D^n(x^k) = [(k!/(k - n)!]x^{k-n}$, we write

$D^a(x^k) = [\Gamma(k + 1)/\Gamma(k - a + 1)]x^{k-a}$

for any real number a. Then if a function can be defined by a Taylor series

$$f(x) - \sum_{k=0}^{\infty} c_k x^k, \text{ we write}$$

$$D^a f(x) = \sum_{k=0}^{\infty} c_k[\Gamma(k + 1)/\Gamma(k - a + 1)]x^{k-a}.$$

The disadvantage of this method is that Γ becomes infinite at nonpositive integers so it does not work for some cases.

Also solved by Mr. Hoylman, Mr. Grant, Mr. Rowe, Mr. Rosenthal, and Donald E. Savage, '54.

9 A few readers were apparently interested in my topology problem:

Let Y be the comb space

$Y = \{(x,y)\epsilon R^2 | 0 \leq Y \leq 1 \text{ and } x = 0, 1/n \text{ for } n \geq 1\}$
$U\{(x,y)\epsilon R^2 | Y = 0 \text{ and } 0 \leq x \leq 1\}$

Prove Y is not contractible relative to (0,1).

The following solution came from Barry J. Yudell, '65:

Let $f{:}Y \times I \rightarrow Y$ be the relative homotopy [i.e., $f(y \times 0) = y$, $f(Y \times 1) = (0,1)$, $f(0,1),t = (0,1)\ t \epsilon I$]. Now $Y \times I$ and Y are metric, and for any n, $f(1/n, 1) \times I$ is a path in Y from $(1/n,1)$ to $(0,1)$. This path P contains the whole "tooth" from $(1/n,1)$ to $(1/n,0)$, for the path is connected, and if it did not contain $(1/n,t)$ for some $t \epsilon I$, then P would be the union of its two open subsets. Let half open seg from $(1/n,1)$ to $(1/n,t)$ be called T; $T \cap P$ and $[Y - (T \cup (1/n,t))] \cap P$, impossible. Now Y (and hence $Y \times I$) is compact, so f is uniformly continuous, and by ϵ,δ considerations, for n large enough, the whole "tooth" could not be contained in $f[(1/n,1) \times I]$ as $f[(0,1) \times I] = (0,1)$.

Also solved by Mr. Hoylman and Mr. Grant.

Better Late Than Never

2 Captain Allan J. MacLaren, '60, has submitted a solution to this problem.

3 I have received a solution to problem 3 from Mr. Zaklad.

79 Although I personally dislike the BASIC system, it appears to have worked for Edward L. Friedman, '50, who submitted the following:

I wrote a computer program in BASIC which solves the problem by trial and error. The answer you published (18,746) was one of many possible correct numbers, the smallest being 3,121. The next five answers are shown on the computer output. Note that the program consists of 11 statements and that execution time is six seconds.

```
UHFD
ON AT  11:28    R MON 12/04/67   TTY 50

USER NUMBER--R28000
SYSTEM--BASIC
NEW OR OLD--OLD
OLD PROBLEM NAME--PLATT
WAIT.

READY.

LIST


PLATT      11:28    R MON 12/04/67

100 LET S=4
110 LET N=5*S
120 FOR I=1 TO 5 STEP 1
130 LET N=(5/4)*N+1
140 NEXT I
150 IF N=INT(N) THEN 180
160 LET S=S+4
170 GO TO 110
180 PRINT "THEY GATHERED"N"COCONUTS"
190 GO TO 160
200 END


RUN


PLATT      11:29    R MON 12/04/67

THEY GATHERED 3121    COCONUTS
THEY GATHERED 18746   COCONUTS
THEY GATHERED 34371   COCONUTS
THEY GATHERED 49996   COCONUTS
THEY GATHERED 65621   COCONUTS
THEY GATHERED 81246   COCONUTS
STOP

RAN      6 SEC.


STOP.
READY.

BYE

*** OFF AT 11:29    R MON 12/04/67.
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81 Also solved by Mrs. Nancy M. Sheehan (wife of Bernard S. Sheehan, S.M. '61), Victor M. Harlick, Mr. Zaklad and John C. Kingery.

Allan J. Gottlieb, '67, is a graduate student in mathematics at Brandeis University. "Puzzle Review" is written for *Technology Review* and *Tech Engineering News*, the M.I.T. undergraduate professional magazine. Mr. Gottlieb's address is in care of the Department of Mathematics, Brandeis University, Waltham, Mass., 02154.