Puzzle Review

Allan J. Gottlieb, '67

Hi. I have been toying with an idea for a rather unusual column for later this year—one in which all the problems would somehow relate to a common theme. Since, according to my girl friend, I live, eat and breathe football, the choice for the common theme is obvious. Hence I would appreciate receiving any football-based problems which you can think of. Although all problems are still welcome, I would prefer that, if possible, you rewrite them in football terms. I realize that girls (except for Tech co-eds, of course) may find this confusing, but maybe next year we'll run one on sewing to even things up.

Now for this month's problems:

Problems

The first three problems (a new record) come from Richard J. Grant, '65:

15 The following example always comes up in elementary number theory: The rationals in [0, 1] have measure zero because if we order them like (0,1,1/2,1/3,2/3,1/4,3/4,1/5,2/5,3/5, . . . 1/n,2/n, . . . (n—1)/n,1/(n+1), . . .) we can cover them with open intervals of length e, (1/2)e, . . . ($1/2^n$)e, so measure of cover is $\leq$ 2e. The problem is: Suppose e = 1/10; then the covering has length $\leq$ 1/5. Exhibit a real number in [0,1] which is not covered.

16 Show that every positive integer n divides some number of the form 111 . . . 1100 . . . 00, i.e., a string of 1's followed by a string of 0's.

17 This one isn't original but doesn't seem as well known as it should be and certainly is more interesting than your topology homework: Show that the intersections of the trisectors of the angles of any triangle form an equilateral triangle.

18 One of my colleagues at Brandeis, Michael R. Gabel, '65, submits the following problem which is an offshoot of some of his work on number theory: Let a be an integer greater than 1. Prove that there exist integers x, y: $1 < x < (2a - 1)$, $1 < y < (2a - 1)$ such that $xy \equiv 2a - 1 \bmod 2a$ if and only if $a > 3$.

19 Last year John A. Maynard, '46, submitted the coconut problem (*see* Technology Review *June, 1967, p. 6).* George H. Ropes, '33, has generalized it to read: N natives gather coconuts. The first native secretly divides the coconuts into N equal piles and has one coconut left over which he discards. He takes one pile and pushes the rest of the coconuts back into one big pile. Native number two does the same with the remaining coconuts, discarding one coconut and taking one pile. This continues through native N. When the last native is through, the number of coconuts left is divisible by N. How many coconuts were gathered in the beginning?

Speed Department

SD5 One of my old friends from Baker House, Chester L. Sandberg, Jr., '67, one of *the* cool guys in the world, submitted a really sharp problem which he read in a Litton Industries ad in *Aviation Week:* Form a limerick out of $(12 + 144 + 20 + 3\sqrt{4}) / 7 + 5(11) = 9^2 + 0$. This is so sharp that I plan to print his solution in three months despite precedents.

SD6 The last problem is from "$(\text{Lin})^2$" Derman: A "prime pair" is two primes whose difference is 2, e.g., 17 and 19. The problem is to prove in less than 30 seconds that the number between any such pair has factor 6, for pairs with smaller prime greater than 6.

Solutions

1 Given the number N, composed of 10 integers, find the number N′, where the first integer is the number of zeros in N, the second is the number of 1's in N, and so forth up to nine, such that N = N′.

I received the following aesthetic masterpiece—beautifully typed— from Edward P. DeLorenzo:

The answer 6,210,001,000 can be shown, by a process of elimination, to be the unique solution. Call the position which describes the number of times the digit n appears the nth position; a number whose digits add to more than 10 is said to be in an overflow condition.

1. The sum of all the digits must equal 10, since these same digits describe the numerical breakdown of the 10 digits.
2. No position can have a 9 because: a. a 9 in the zero position implies a 0 in the nine position; and b. a 9 in any other position implies overflow.
3. No position can have an 8 because: a. an 8 in the zero position implies a 0 in the one postion and a 1 in the eight position; and b. an 8 in any other position implies overflow.
4. No position can have a 7 because: a. a 7 in the zero position implies a 1 in the seven position, a 0 in the two position, and a 1 or a 2 in the one position; and b. a 7 in any other position implies overflow.
5. There must be at least a 3 in the zero position since the seven, eight, and nine positions must have 0's.
6. The largest digit possible in the one position is a 2: a. a 3 in the one position yields a sum of nine and the 2 required in the three position implies overflow; b. 4 and 5 imply overflow; and c. 6 is impossible.
7. Neither the two nor three position can have a number larger than 3: a. 4 and 5 imply overflow; and b. 6 is impossible.
8. The four, five, and six positions are limited to the digits 0 or 1; any larger digit can be shown to imply overflow.
9. There must be a 6 in the zero position: a. a 3 implies 0's in four, five, and six, which is a contradiction; b. a 4 implies 0's in five and six, also a contradiction; c. a 5 implies 0's in positions four, six, seven, eight, and nine and a 1 in the five position which implies a 2 in the one position. This leaves the two and three positions to be filled: the sum of the digits in two and three must be two; neither can have a 0 since all five 0's are accounted for (this implies neither can have a 2 or a 3); and finally a 1 in each of these positions implies a contradiction.
10. There must be a 1 in the six position and 0's in positions four, five, seven, eight and nine.
11. The remaining 0 is in the three position; a non-zero digit in the three position implies a 3 in either position one or two, both of which are impossible.
12. There is a 2 in the one position; 0 or 1 implies a contradiction.

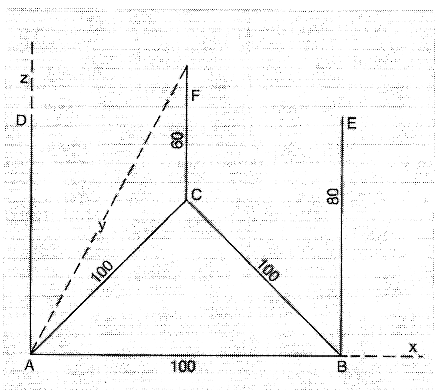
13. There must be a 1 in the two position for the sum of the digits to equal 10.
14. The steps to this point show that if a number exists which satisfies the conditions it must be 6,210,001,000. An examination of the number shows that it in fact does satisfy the conditions.

Also solved by Mark H. Yu, '70, Douglas J. Hoylman, '64, Peter E. Lobban, '66, Eric Rosenthal, E. Alan Phillips, '57, John L. Joseph, '40, Beverly Seavey, and Abba M. Krieger, '69.

2 A flat triangular field, 100 feet per side, has 60-, 80-, and 100-foot-high flag poles at the respective corners. Determine the length and base location for a step ladder that can be rotated and just touch the top of each flag pole.

Mr. Lobban sent in the following solution:

The problem bows to analytical geometry. Let the triangle be placed in the x,y plane of a three-dimensional coordinate system as shown, with AD || BE || CF || z-axis.



The coordinates are as follows:

A: (0,0,0)
B: (100,0,0)
C: $(50,50\sqrt{3},0)$
D: (0,0,100) top of 100′ ladder
E: (100,0,80) top of 80′ ladder
F: $(50,50\sqrt{3},60)$ top 60′ ladder

The problem is to find a point $0 = (x,y,z)$ such that $\overline{OO} = \overline{OE} = \overline{OF}$ and O is in the plane of the triangle. Clearly $z = 0$ because of this latter condition. Then $(x - 0)^2 + (y - 0)^2 + (0 - 100)^2 = (x - 100)^2 + (y - 0)^2 + (-80)^2$, for $\overline{OD} = \overline{OE}$ implies $\overline{OD}^2 = \overline{OE}^2$.

Similarly,
$(x - 100)^2 + (y - 0)^2 + (-80)^2 = (x - 50)^2 + (y - 50\sqrt{3})^2 + (-60)^2$.
The first of these equations reduces to $x^2 + y^2 + 100^2 = x^2 - 200x + 100^2 + y^2 + 6400$, whence $200x = 6400$ or $x = 32$. Plugging this into the second gives $68^2 + y^2 + 80^2 = 18^2 + y^2 - 100\sqrt{3}\,y + 7500 + 3600$ or $100\sqrt{3}\,y = 400$ and $y = (4\sqrt{3})/3$.

The length of the ladder is, of course, $\overline{OD}$ or $\overline{OE}$ or $\overline{OF} = \sqrt{x^2 + y^2 + 100^2} = \sqrt{10000 + 1024 + 16/3} = 8/3\,(\sqrt{1551})$.

Also solved by Mr. Rosenthal, Mr Yu, Mr. Hoylman, Norman M. Wickstrand, '29, John E. Prussing, '62, Jeffrey Robinson (from John W. McNear, '59), F. Wade Greer, Jr., '52, and Douglas K. Severn, '23.

3 The problem:

	♠ A 7 3 2 ♥ A 7 2 ♦ J 9 5 ♣ A K 10	
♠ K J 9 6 4 ♥ K J 8 6 5 ♦ 7 2 ♣ 5		♠ Q 8 5 ♥ Q 9 4 3 ♦ — ♣ Q J 9 6 3 2
	♠ 10 ♥ 10 ♦ A K Q 10 8 6 4 3 ♣ 8 7 4	

Contract: seven diamonds by South; opening lead: club five.

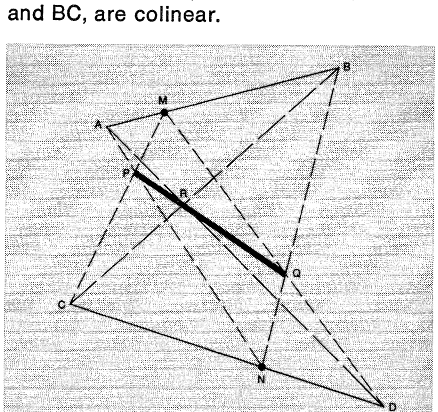
I received the following letter from Warren Himmelberger:

A squeeze is needed to develop the 13th trick. The best chance is to find East with no more than three spades so that the ♠7 can be used to squeeze West. After this bit of analysis, the hand plays easily.

Take the opening lead with the ♣ A. Play two rounds of trumps and lead a club to ♣ K. Play the ♠ A and ruff a spade. Lead a low trump to dummy's ♦ J and ruff a third spade. Play two more trumps, discarding dummy's ♣ 10 and ♥ 2. South then leads his last trump, squeezing West. If West drops his ♠ K, dummy's ♠ 7 is good. So West discards the ♥ J. Now dummy discards the ♠ 7 and East is squeezed. If East discards his ♣ J, South's ♣ 4 will be good. East throws the ♥ 9. A heart to dummy's ♥ A drops the ♥ K and ♥ Q, and the ♥ 7 takes the 13th trick.

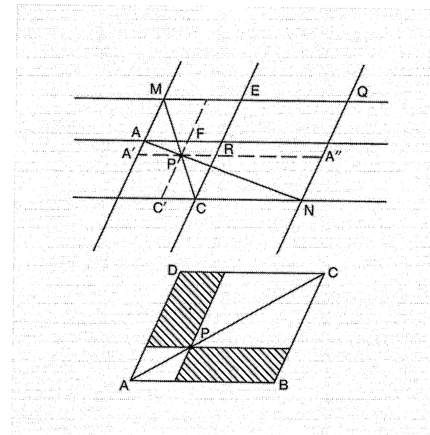
Also solved by Mr. Severn, Allen L. Zaklad, '65, George J. Marlowe, Sc.D.'50, Alvan L. Davis, '98, Charles D. Coltharp, '58, and Victor M. Harlick.

4 Given any two segments AB and CD and M is any point on AB and N is any point on CD, show that intersection P of MC with AN and intersection Q of MD and NB and R, the intersection of AD and BC, are colinear.



Mr. Yu submitted the following solution:

This problem may be handled by projective geometry. Let the projective point source be on the same level as the line BD. We have then, using the same letters as in the original diagram:



For convenience, other points on the diagram are arbitrarily labeled; through P draw A′A″ parallel to MQ and PC′ parallel to AM. Now to prove that P, R, and C are colinear, it is necessary to state a theorem of Euclid: If ABCD is a parallelogram, and AC is a diagonal, then we may pick any point P on AC such that the unshaded parallelograms have equal area. The following equalities are then obvious: Parallelogram A′C′ = parallelogram PE and parallelogram A′C′ = parallelogram A″F. Hence parallelogram PE = parallelogram A″F, or parallelogram EF = parallelogram RA″. This means that P lies on the diagonal RQ. Q.E.D.

Also solved by Mr. Rosenthal.

5 Express the volume of a regular dodecahedron in terms of the length of an edge.

The neatest solution comes from Mr. Severn, with a two-dimensional section.

Let the length of one side equal a. The volume V equals the surface area A of the solid times 1/3 the distance d to the center; and the surface area equals 12 times the surface of one pentagonal face. Therefore,
$A = 12\,[a/2 \times a/2 (\cos 36/\sin 36) \times 5]$
$= 12\,xa^2/4 \times 5 \times 1.3764 = 20.646\,a^2$.
$D = (a + x + y) \cos 30$
$= 2.5387 \times \sqrt{3}/2 \times a$
$= 2.1985\,a$.
$d = D/2 = 1.0963\,a$; $d/3 = .3654\,a$.
$V = 20.646\,a^2 \times .3654\,a = 7.544\,a^3$

Also solved by Mr. Yu, Mr. Rosenthal, Mr. Hoylman, and Norman L. Apollonio, '22.

Allan J. Gottlieb, '67, is a graduate student in mathematics at Brandeis University. "Puzzle Review" is written for *Technology Review* and *Tech Engineering News,* the M.I.T. undergraduate professional magazine.