Puzzle Review

Allan J. Gottlieb, '67

Hello again. I should like to greet any old-timers who are back for another year of frustration. Of course new readers with fresh ideas (and fresh problems) are also welcome; I'll explain the ground rules to them.

Each month I shall present between five and seven problems of varying difficulty. Some will be so easy that even I can do them, and for some I will have no solution available as of press time. The latter will be denoted by having a ♦ to the left of the problem number. Two months after the problems appear, the solutions will be printed. I prefer to use problems that my readers send in, but, in an emergency, I have access to some of my own.

All correspondence should be addressed to:

Allan Gottlieb
Department of Mathematics
Brandeis University
Waltham, Massachusetts 02154

Yes, Virginia, that's Brandeis. My official academic connection with M.I.T. ended in June when I received my bachelor's in mathematics. I am now in the *green* land of Waltham admiring the *trees* and working towards a Ph.D.

O.K., let's get to the problems.

Problems

1 The following letter was sent in by John H. Boynton, '58:

Dear Allan,
Your column does as it is intended—provides relaxation while provoking logical thought. Most of my math, as I am sure is the case with other seasoned alumni, is too far behind me to solve the bulk of your interesting problems. The graphical and trial-and-error puzzles, however, have given me much enjoyment. Keep up the good column.

I would like to propose a problem: Given the number N, composed of 10 integers, find the number N′, where the first integer is the number of zeros in N, the second integer is the number of ones in N, and so forth up to nine, such that N = N′. This puzzle can obviously be solved by trial and error, which is the way I did it, but you might ask your readers to develop an analytical technique, if one exists. The answer I got is 6,210,001,000. I cannot testify to the uniqueness of the answer, since I stopped when I found one solution, and without an analytical representation, I cannot prove it.

2 F. Wade Greer, Jr., '52, proposed the second problem for this issue.

How about the one about the flat triangular field, 100 feet per side having 60-, 80-, and 100-foot-high flag poles at the respective corners. The problem is to determine the length and base location for a step ladder that can be rotated and just touch the top of each flag pole.

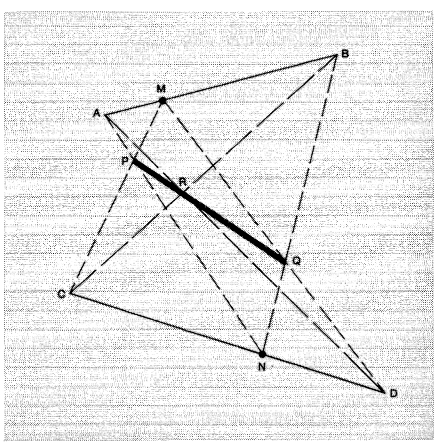
3 One of my friends, Fred Ciaramaglia, '69, has supplied the following bridge problem:

	♠ A 7 3 2 ♥ A 7 2 ♦ J 9 5 ♣ A K 10	
♠ K J 9 6 4 ♥ K J 8 6 5 ♦ 7 2 ♣ 5		♠ Q 8 5 ♥ Q 9 4 3 ♦ — ♣ Q J 9 6 3 2
	♠ 10 ♥ 10 ♦ A K Q 10 8 6 4 3 ♣ 8 7 4	

Contract: seven diamonds by South; opening lead: club five.

4 John L. Joseph, '40, a "Puzzle Review" regular, sends in the following challenge:

Given any two segments AB and CD and M is any point on AB and N is any point on CD, show that intersection P of MC with AN and intersection Q of MD and NB and R, the intersection of AD and BC are colinear.



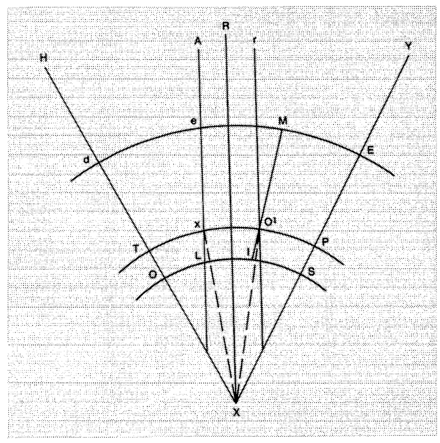
5 The last regular problem is proposed by an army captain, Roger A. Whitman, '61: Express the volume of a regular dodecahedron in terms of the length of an edge.

(O.K., new math types, let's go—Ed.)

Speed Department

(These problems are less rigorous and answers are not printed—Ed.)

SD1 What's wrong with Harry Dee's ('67) solution to the impossible problem of trisecting an angle with straight edge and compass?



Bisect the given angle HXY with RX. Draw parallels AL and rO¹ at equal distances to RX. With X as center, swing arcs OS and dE. Bisect arc LS at I, and arc eE at M. Describe arc TP through O¹, where MI cuts the parallel. The angle is trisected at x and O¹, because MI is the locus of the midpoints of all the arcs drawn with X as center and contained between AL and XY. QEF and QED.

SD2 Sanford S. Miller, '60, wants to know what is the next term in the sequence O,T,T,F,F,S,S,E, .

Solutions

78 Given: 11 balls of same weight plus one heavier or lighter (12 total). Identify the odd ball and determine whether it is heavier or lighter with three weighings on a two-pan balance. To make it more difficult, what is the maximum number of balls from which the odd one can be identified with four weighings? with five weighings? with n weighings?

The "neatest" solution to the problem was submitted by Doug Hoylman, '64, who typed his response.

Letter the balls ABCDEFGHIJKL. For the first weighing, balance ABCD against EFGH.

I If it balances: The "bad" ball is one of IJKL. For the second weighing, match IJ against KA.

1. If this balances, L is the bad ball, and it may be matched against any "good" ball for the third weighing, to determine if it is heavy or light.
2. If IJ is heavier, then either I is heavy, J is heavy, or K is light. Match IK against any two good balls for the third weighing. If IK is heavier, I is heavy; if IK is lighter, K is light; if it balances, J is heavy.
3. If IJ is lighter, interchange "heavy" and "light" throughout 2.

II If the first weighing does not balance: Without loss of generality we may assume ABCD is heavier. Then either one of ABCD is heavy, or one of EFGH is light. For the second weighing, match ABE against CDF.

1. If this balances, then either G or H is light, and they may be matched against each other for the third weighing.
2. If ABE is heavier, then either A is heavy, B is heavy, or F is light. For the third weighing, match AF against any two good balls. If AF is heavier, A is heavy; if AF is lighter, F is light; if it balances, B is heavy.
3. If CDF is heavier, replace A by C, B by D, and F by E in the above instructions.

For the more general question of how many balls can be thus judged by n weighings, let m be the maximum such number. Then for m balls, there are 2m possible situations. For each weighing there are three possible outcomes (left side heavy, right side heavy, balance), hence for n weighings there are 3^n possible outcomes, and these must distinguish between the 2m situations, so we must have $2m \leq 3^n$, or $m \leq 3^n/2$, or, since 3^n is odd and m is an integer, $m \leq (3^n - 1)/2$. This gives an upper bound for m, but this need not be attained for any particular n. In particular, for $n = 3$ this formula gives $m \leq 13$, while we actually have $m = 12$.

Also solved by John Joseph, Sanford Miller, and Shih-Ping Wang, '61, who included two similar problems which will appear next time.

79 Five island natives spend all day gathering coconuts. They finish the job when it is dark and decide to leave the coconuts in a pile and come back in the morning to divide them up. At midnight one native wakes up and decides he wants his 1/5 now. He goes to the pile, divides it into five equal piles, and finds that there is one coconut left over. He gives it to a monkey. The native takes away one of the five piles, leaving the other four piles there. He pushes the four piles into one big pile as he leaves. At 1 a.m. native #2 wakes up and decides he wants his 1/5. He goes to the pile, divides it into five equal piles, finds one coconut left over, gives it to the monkey, takes away one pile, and leaves the rest. At 2 a.m. native #3 does the same thing: divides the total into five equal piles, gives one nut left over to the monkey, takes away one pile. At 3 a.m. native #4 does the same thing. At 4 a.m. native #5 does the same thing. After native #5 takes his coconuts away, the quantity of nuts left is evenly divisible by five. How many coconuts were gathered by the natives?

John Joseph sent in the following solution:

left	taken	native
n	$(n - 1)/5$	1
$(4n - 4)/5$	$(4n - 9)/25$	2
$(16n - 36)/25$	$(16n - 61)/125$	3
$(64n - 244)/125$	$(64n - 369)/625$	4
$(256n - 1476)/625$	$(256n - 2101)/3125$	5

$(1024n - 8404)/3125 = 5p$ (p is an integer)
$(15625p + 8404)/1024 = n.$

p must be divisible by 4. Let $4r = p$.
$(15625r + 2101)/256 = n$
$61(9/256r) + 8(53/256) = n$
$(9r + 53)/256 = n - 8 - 61r = s$
$9r = 256s - 53.$

By the sum of digits, $4s - 8$ must be divisible by 9.
Try $s = 11$:
$9r = 2816 - 53 = 2763$
$r = 307$

$p = 1228$

$n = 11 + 8 + (61 \times 307) = 18{,}746.$

Also solved by Eric Rosenthal, son of Meyer S. Rosenthal, '47, Paul J. Schweitzer, '61, Jon Livingston, John E. Prussing, '62, Warren J. Himmelberger, '47, Vernon J. Wyatt, Doug Hoylman, and F. Wade Greer, Jr., who adds "John A. Maynard, '46, is a stinker! Had a lousy weekend!"

81 All four hands may be viewed in this bridge problem:

	♠ 5	
	♥ 8 5	
	♦ A K 7	
	♣ A K 8 6 4 3 2	
♠ K 10 7		♠ 8 6 4 3 2
♥ 9		♥ Q 7 6 2
♦ Q 10 8 3		♦ J 6 2
♣ Q J 10 9 7		♣ 5
	♠ A Q J 9	
	♥ A K J 10 4 3	
	♦ 9 5 4	
	♣ —	

Bid 7 hearts by South. Lead queen of clubs. Problem: make bid against *any* defense (after the club lead).

This problem caused a great deal of interest. Several people wrote to me over the summer asking for a solution. Since there would have been a several month delay until the answer appeared, I complied with their requests. This will not be a general practice. The solution I sent to them was by Peter J. Davis, Jr., son of Peter J. Davis, '48, and is as follows:

A double finesse could drop East's queen of hearts — but only at the expense of losing a spade trick to West. So a spade trick must be promoted while East's holding is pared down until he is forced to ruff a lead from the board, giving South his final trump finesse. West's club lead is taken on the board, and South sluffs a diamond. A heart is led from the board; East can't help but play low, and South finesses the 10. The ace of spades is cashed, and the jack of spades is led toward the board. If West plays high, the declarer ruffs; if low, a low club is sluffed and another spade is led. West's king of spades drops. South ruffs on the board, and a low club is returned. If East ruffs, South overruffs, draws trump and has no problem cashing his winners to make the contract. If East sluffs—probably a spade—South ruffs, leads a diamond to the board and ruffs another club. The last diamond is led to the ace on the board, and the ace of clubs is led. If East ruffs, South overruffs, draws trump, and he holds a good spade trick. If East sluffs, South sluffs his spade and any card may be led from the board; East has to play a trump, and South gets his final finesse. South's remaining trumps are good.

Another elegant solution was submitted by James Kaltenbronn, Ph.D. '60.

82 The unit disk E^2 is defined as the set of all points (x, y) such that $x^2 + y^2 < 1$ (x and y are real numbers). Prove the Brouwer fixed point theorem which asserts that every continuous function f from E^2 to E^2 has a fixed point, i.e., there is an x in E^2 such that $f(x) = x$.

Doug Hoylman found a typographical error in the statement of the problem. He states:

I hope none of your readers worked too hard on this one, because as stated (for the open disk) it's false. For a counterexample, superimpose on the plane a polar co-ordinate system with origin at (—1, 0), and consider the function $(r, \theta) \rightarrow (r/2, \theta)$. Clearly this is continuous, but it has no fixed point, since $r = 0$ is now in the open disk. The Brouwer fixed point theorem is stated for the *closed* disk, $x^2 + y^2 \leq 1$ (and can be generalized to any convex compact subset of the Banach space). I don't know how to prove it. For a quick problem, your readers might try to prove the corresponding result in one dimension, i.e., any continuous function from (—1, 1) to itself has a fixed point.

(For a proof of the Brouwer fixed point theorem, see Homology Theory *by P.J. Hilton and S. Wylie. — Ed.)*