Puzzle Corner

By Allan J. Gottlieb, '67

Hi. With the pressure of grad school acceptance finally off my shoulders, I can start to sleep at night and even smile once or twice during the day. Adding to my happiness is the wonderful response you are giving my column. I am afraid that the problems for this issue may be of lower quality since several of them are mine.

Problems

68—The first puzzle was submitted by Stuart P. Keeler, '57.

I would like to give you the following puzzle for your "math experts" to work on. I think they will have lots of fun trying to tackle this one. It is my favorite. By the way, my undergraduate degree at M.I.T. was in Course IX—General Engineering.

Insert the remaining digits 7, 8, and 9 into the proper place in the following sequence:
—, 5, 4, —, 1, —, 6, 3, 2, 0.

69—Arrange 21 points so that they form 12 lines each having precisely 5 points.

I have been asked to include an occasional mathematical problem so here is one from my midterm exam in graduate algebraic topology.

70—Find a relation among the betti numbers of the homology of a compact orientable non-bounded n-dimensional manifold. Do the same for the corresponding torsion subgroups. A hint was given: compute many "simple examples." Proofs were *not* required.

71—Thomas F. Hickerson, '09, would like to know the minimum integral values of A, B, and C such that $A(A + 8) = B(B + 28) = C(C + 34)$.

72—Arthur Mohan, '08, sent in the following problem:

In how many ways may "m" different blue books and "n" different red books be arranged on a shelf so that no two of the red books are together? What inequality must "m" and "n" satisfy? This is supposed to be solved without the use of any theory of groups or sets.

73—Show that if $2^p - 1$ is a prime then $2^{p-1}(2^p - 1)$ is perfect.

Speed Department

The only entry for this month comes from Arthur D. Delagrange, '61, who asks:

74—Why does Technology Review always arrive the night before a quiz?

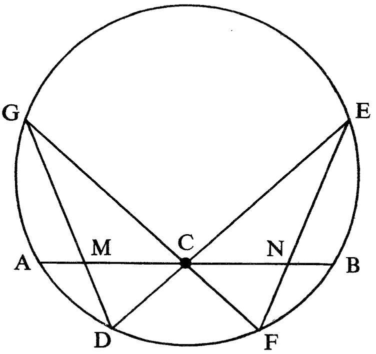
Solutions

28♦—1 is the smallest integer that is a perfect cube, a perfect 4th power, and a perfect 5th power. What is the next smallest integer with this property? Any number raised to the 60th power has the property, but surely there is some number smaller than 2^{60}.

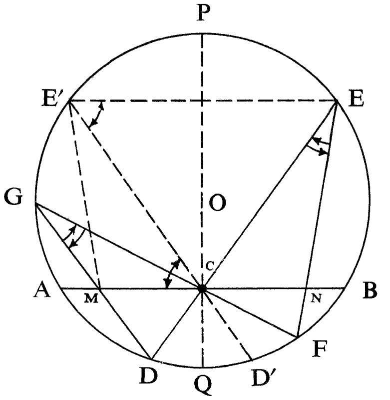
E. Alan Phillips, '57, has shown that the smallest such integer must be 2^{60}. This number must be factorable in the form $P_1^{n_1} P_2^{n_2} P_3^{n_3} \ldots$, and hence $n_1, n_2, n_3, \ldots$ must each be divisible by 3, 4, and 5—that is, if nonzero they must be divisible by 60. Therefore 2^{60} is the answer.

Also solved by Douglas J. Hoylman, '64, John L. Joseph, '40, and William P. Bengen, '69.

29♦—Given a circle, Q, with a chord AB and C its midpoint. Through C draw any two chords DE and FG. Form DG and FE intersecting AB at M and N respectively. **Prove MC = NC.**



Mark H. Yu, '70, my analytic geometry expert, came through again. Mr. Joseph gave a more geometrical argument which I shall use.



Label center O. Extend diameter OC to P and Q. Draw EE′ ⊥ to PQ. Extend E′C to D′. Join GE′.

Angle E′GM = ½ arc E′PEFD = $\dfrac{180° + E'P + QD}{2}$

Angle E′CM = <EE′M since EE′ || AB

Angle E′CM = ½ arc EBFD′ = $\dfrac{180° - \overset{\frown}{PE} - \overset{\frown}{D'Q}}{2}$

since $\overset{\frown}{E'P} = \overset{\frown}{PE}$ and $\overset{\frown}{D'Q} = \overset{\frown}{QD}$

Angles E′GM + E′CM = 180°

∴ points E′, C, M and G are inscribable in a circle and

∴ Angle ME′C = angles MGC + CEN

Compare △ ME′C with △ NEC

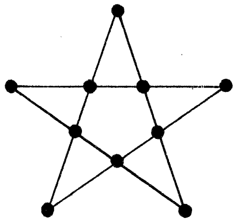
Angle ME′C = angle NEC; E′C = EC;

Angle E′CM = angle ECN

∴ they are congruent.

30—Arrange 10 dots on a piece of paper (*"10 points in* [*0,1*] x [*0,1*]" *for you mathy types—ed.*) in such a way that five line segments can be drawn through the dots, and each line segment contains exactly four dots. No two line segments can be colinear.

This was so easy that the following solution was submitted by Allan J. Gottlieb, '67. When he solves them they really must be trivial.



Also solved by Douglas Hoylman, John W. McNear, '59, Alan S. Ratner, '69, Peter L. Eirich, '69, A. Ostapenko, Richard D. Minnich, '65, Steven L. Oreck, '70, Larry Horton, Erich S. Kranz, '70, John Joseph, and Mark Yu, who found two slightly different but exceedingly ugly shapes.

31—Let C be the set of those reals whose base 3 expansions have no 1's. Can every real be expressed as a finite sum of elements of C?

This one really caused a stir. Two people claimed that the answer is no because they only considered whole numbers. Mr. Phillips sent in a proof valid only for the rationals. Douglas Hoylman, who has already won a free subscription, sent in the following:

Okay, here it is, the first correct answer: Yes.

Seriously, I think I can show this using the fact that the Cantor set is closed and things like that, but I don't want to hog *all* the free subscriptions. If you desperately need a solution I'll send you mine.

This time, however, I am not desperate and must award two new prizes.

The "M.I.T. Freshman" strikes again! A little thought yields an answer in under 15 minutes to problem 31.

It is obvious that every C-number can be expressed in the form:

$$C = \sum_{i=-\infty}^{\infty} C_i 3^i \quad (C_i = 0 \text{ or } 2)$$

Define the class of K-numbers by

$$\{K\} = \left\{\frac{C}{2}\right\}$$

Each K-number will be of the form

$$K = \sum_{i=-\infty}^{\infty} k_i 3^i \quad (k_i = 0 \text{ or } 1)$$

It is apparent that every real can be expressed as the sum of two K-numbers. Therefore, given a real to be expressed by $C_1 + C_2$, define K_1 and K_2 such that

$K_1 + K_2 = (C_1/2) + (C_2/2) = R/2$

Multiplying the above by 2, we obtain $C_1 + C_2 + R$. Q.E.D.

—Paul Hughett, '70, East Campus