# Puzzle Corner

By Allan J. Gottlieb, '67


I have been so deluged with problems and solutions that I must actually save many for next time. But don't worry—eventually all will appear. And don't be discouraged—write to me at Box 4380, Baker House, 362 Memorial Drive, Cambridge, Mass., 02139.

## Problems

21—First we have the following letter from Nicholas J. Pippenger, '67:

Dear Mr. Gottlieb:
It is, I feel, ironic that *Tech Engineering News* should feature a "Puzzle Corner" which inclines toward mathematical diversions. In an effort to correct this situation I offer the following original problem which, as far as I know, has never appeared in print:

A magnetic dipole $m\bar{I}_z$ is situated at the origin of cylindrical coordinates $(r,\phi,z)$. A charge of q is situated at (2,0,0). In a situation such as this it is well known that the Poynting vector does not vanish so there is an energy flux $\bar{S} = (1/\mu_0)E \times B$ and a momentum $\bar{P} = \epsilon_0 \bar{E} \times \bar{B}$ even though the configuration is entirely static (see, for example, Feynman's *Lectures* Volume II, Chapter 27). The problem is to find the total angular momentum of the electromagnetic field about the z axis; that is, find the integral

$$L_z = \int r\bar{I}_\phi \cdot (\epsilon_0\bar{E} \times \bar{B})\, dV$$

over all space. (Assuming q, m, and $\epsilon$ finite and nonzero, then $L_z$ will be finite and not zero).

22a—Don B. Zagier, '70, included a puzzle this time in addition to solving all the problems. He would like to know, given a regular n-gon $A_1A_2 \ldots A_n$ having the property that $1/A_1A_2 = 1/A_1A_3 + 1/A_1A_4$, what is n?

♦23—Doug Hoylman, '64, adds:

Here's an interesting problem that's not too difficult: Prove that

$$\sum_{n=1}^{\infty} 3^{-n!}$$

converges to an *irrational* number. (The solution can be expressed in five words. This is one of the so-called Liouville numbers and can be shown rather easily to be transcendental.)

24—My favorite puzzle is the following sent in by Paul W. Abrahams, '56:

Find the missing element in the following sequence (whose rightward continuation is undefined):

10, 11, 12, 13, 14, 15, 16, 17, 20, 22, 24, __, 100, 121, 10000.

♦25—The following challenge comes from Frank G. Smith, '11:

I have a problem that no one seems to want to tackle or even acknowledge: Two ladders 30 and 40 feet long are set between two vertical walls, the tops of the ladders on opposite walls. They cross at 10 feet above the horizontal between the walls. Question: How far apart are the walls?

This is not so easy as it looks. It requires some wangling with right triangles and it finally simmers down to a fourth power (diminishing power) equation to solve "a" in "b" and then get an answer. I can solve it only by trial and error to what I think is (in feet) accurate to about .01 per cent.

If you like my problem, I'll send you my method of solving it. But I expect you young mathematicians would have a better, more accurate way.

*Let's see.*

## Speed Department

26—I received the following letter:

Dear Mr. Gottlieb:
I enjoyed your puzzle corner in the Technology Review for November, and will accept your invitation to submit puzzle suggestions.

There are three automobile tags. The first has two digits, the second has three digits, the third five digits. The number on the first multiplied by the number on the second equals the number on the third. The second divided by the first gives a whole number. All the digits from 0 to 9 appear on the tags. What are the numbers on the tags?

Yours truly,
Hubert duB. Lewis, '37

27—The final problem comes from Alvan L. Davis, '98:

Interpret the following sequence: "H I J K L M N O."

## Solutions

8—Prove that for any even integer m greater than 2, there is an infinity of odd integers not the sum of a prime and a positive power (> 1) of m.

Don B. Zagier, '70, came close:

I can prove this only for $m > 6$, but in a more general fashion: the power of m can be 0 or 1 as well as > 1. The argument is probabilistic (*i.e., it is probably valid—Ed.*). Assume that every odd number $> n_0$ is the sum of a prime and a power $(1, m, m^2, \ldots)$ of m. Consider $n > n_0$. There are $[(n - n_0)/2]$ odd numbers z with $n \geq z > n_0$ (the [ ] is the greatest integer function). There are p(n) primes $\leq n_0$ (p(x) is the number of primes $\leq x$). There are $[\log_m n] + 1$ powers of $m \leq n$. Hence there are at most $p(n)([\log_m n] + 1)$ numbers of the form $p + m^k$, where $p \leq n$, $m^k \leq n$. Hence $[(n - n_0)/2] \leq p(n)([\log_m n] + 1)$, so $n/2 - (n_0 + 1)/2 \leq [(n - n_0)/2] \leq p(n)([\log_m n] + 1) \leq p(n)\log_m n + p(n)$ or $1/2 - (n_0 + 1)/2n \leq p(n)\log_m n/n + p(n)/n = p(n)\log_e n/n\log_e m + p(n)/n$. As $n \to \infty$, $p(n)/n \to 0$, $p(n)\log_e n/n \to 1$, so we have $1/2 \leq 1/\log_e m$, or $\log_e m \leq 2$, $m \leq e^2$. Since m is an even integer, m = 2, 4, or 6. Hence for $m > 6$, it is false that an infinity of odd numbers are representable as $p + m^k$ with p prime, m even, $k \geq 0$.

For the following I am indebted to the

Mathematical Association of America: With $k > 0$, the odd number $m(m-1)^k + 1$ may not be written as $p + m^b$, p a prime, $b \geq 1$. Recalling that $m > 2$, it follows from $m(m-1)^k + 1 = p + m^b$ that $p \equiv 1 \pmod{m}$ and that $(m-1)$ divides p, i.e., $p = m - 1$, an impossible situation.

9—Show that there are irrational numbers s and t such that $s^t$ is rational.

Check out the following:

**Kid:**
**#9 is $e^{\ln 2} = 2$**
Courtesy of the Green Phantom.

Also solved by Mr. Zagier and Peter Groot, '68.

10—Assuming

$f(n) = \sqrt{n + \sqrt{n + \sqrt{n + \dots}}}$

converges for all integers n, show that given any integer y there is an integer n such that f(n) converges to y.

Identical proofs were submitted by Mr. Zagier and Mark Yu, '70:

$f(n) = \sqrt{n + \sqrt{n + \sqrt{n + \dots}}} = y$, by convergence $\therefore \sqrt{n + y} = y$ or $n = y^2 - y$. Q.E.D.!

Mr. Groot also solved this problem and adds that y must be 0 or greater than 1.

11—For which positive values of a and c is $a^n \cdot n! > c \cdot n^n$ true for every positive integer n?

Mr. Yu's proof is the most direct, but Mr. Zagier's is slicker (i.e., shorter): and so I will print his:

**Take $n = 1$. Then $a^1 \cdot 1!$ must be $> c \cdot 1^1$, or $a > c$. To prove by induction, assume $a^k \cdot k! > c \cdot k^k$. Then $a^{k+1} \cdot (k+1)! = a(k+1) \cdot c \cdot k^k = a(k/(k+1))^k < (k+1)^{k+1}$. If this is $> c(k+1)^{k+1}$, a must be $> ((k+1)/k)^k$. This is true if $a \geq e$. Hence, if $a > c$, $a \geq e$, then $a^n \cdot n! > cn^n$ for all n.**

12—What is the largest number of queens which can be placed on a chess board such that no three queens lie in a straight line. Any solution greater than 14 will be printed.

Chesley E. Osborn, '67, Hugh J. Vishner, '70, and Mr. Zagier each sent in one solution with 16 queens, but Mr. Groot sent in 72 solutions.

I have received several letters since my January column went to the printer; thus I can only mention their names here. Eric Rosenthal, ('73?) solved problems 1, 3, 4, 6, and 7. (By the way, Eric, your solution to 6 was cool.) Eric's father, Meyer S. Rosenthal, '47, solved 5. What a family! Why didn't you get Mom to tackle 2? Paul W. Abrahams, '56, solved 1, 2, 4, and 6. Frank G. Smith, '11, sent to me a solution to problem 6 plus a Christmas card; thank you. James H. Michelman, '51, solved 1. Problem 5 was successfully attacked by M. E. MacGregor, '07, and number 2 fell to Paul Langacker, '68, who reads TR, not TEN.