Puzzle Corner

By Allan J. Gottlieb, '67

Thank you, one and all. Your many letters containing problems and solutions, when coupled with random threatening letters from my creditors, have caused my mailbox to be continually full. This has done more for my persecution complex than could possibly be accomplished by a \$20-visit to a medic.

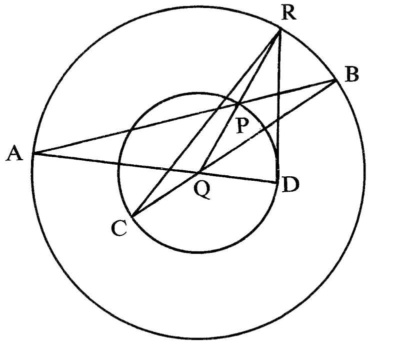
Please send your solutions to me addressed to Baker 4380, 362 Memorial Drive, Cambridge, Mass. 02139.

Problems

15—Place 7 unbent cigarettes such that each one is touching the other 6.

16♦—Prove that none of the Platonic solids may be dissected into another (two-page proof). *Submitted by Mark H. Yu, '70.*

17♦—Prove that $\overline{RD}^2 - \overline{RC}^2 = \overline{PB}^2 - \overline{PA}^2$.



The circles have center Q, and CB and AD intersect at Q. *Submitted by Mark H. Yu, '70.*

18♦—I have received the following letter:

Dear Mr. Gottlieb:
Last year at Wilson High School in Washington, D.C., one of my classmates, Michael Reedy (now at the University of Chicago) developed the Storey Intelligence Integral (named after Thomas Storey, now at the U.S.N.A.). The following is an example:

Evaluate in five seconds or less:

$$\int_{-e^{-e^2}}^{\sin^3 4} \int_{\int_{-2}^{2}(y^3+3)\,dy}^{25\cosh(\ln 3)} \int_{-e^{-\ln(1/2)}}^{\int_{\pi}^{2\pi}\sin X\,dX} \frac{(\cosh^2(x+2)\sin x^3)^{x^2-x}\,dx}{\ln(e^{2x}+\ln 2x^{e^x}-1)^{x+2\sin x}}$$

Mark B. Pelcovits, '70

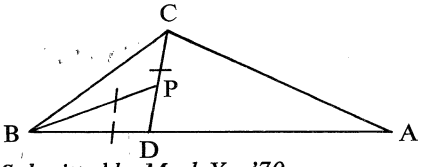
I will accept any evaluation performed in 5 *weeks* or less.

The Speed Department

19—I have received the following suggestion for a speed problem from Alan S. Ratner, '69:

$$\lim_{2\to\infty}\left(\frac{1}{2}\right) = \_\_\_\_\_$$

20—Prove that it is impossible to construct a point P in the general triangle ABC such that CP=BP=BD.



Submitted by Mark Yu, '70.

Solutions

1—The problem:

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  W A S M A R C H
+    T H E B E S T
C A N D I D A T E
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The product H•E•R•B•I•D equals 0. And M, which does not equal zero, equals C•W.

The following was contributed by Don B. Zagier, '70:

We consider only the case where W, T, and C are nonzero (i.e., the numbers are real 8-, 7-, and 9-digit numbers). This is not a partial solution: since I produce 2 solutions, though the problem was supposed to have a unique solution, it is the problem rather than the solution that fails to be exhaustive. The solutions are

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   90790912            90790813
 +  9219179    and   +  9320279
  100010091           100111092
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Method: Plainly, W is 9, C is 1, A is 0. Hence M=CW=9. The third column then gives T is 9, N is 0. E=0 leads to a contradiction, so the seventh column gives R+E=10, D=B+1 mod 10. H=0 gives a contradiction, so from the last column H=E+1, so S is 7 from the 8th column. From H·E·R·B·I·D=0 we get H=E+1=D+2. There are now two possibilities from the 6th column: D=0, B=9 (which gives first solution) or B$\neq$9, D=B+1, D=I, R+D=9, and R or B=0 (from H·E·R·B·I·D=0) so (since the 4th column gives D $\leq$ 7) D$\neq$9, R$\neq$0, so B=0, D=I=1, R=8, and we have the second solution.

David R. Spencer, '63, also submitted both solutions. The latter was also provided by Robert H. Parker, '70, Richard Bator, '65, and William W. Upham, '23; and the former by Kalman Shure, '51, John L. Joseph, '40, and Mark H. Yu, '70, who also found a completely different solution, "No."

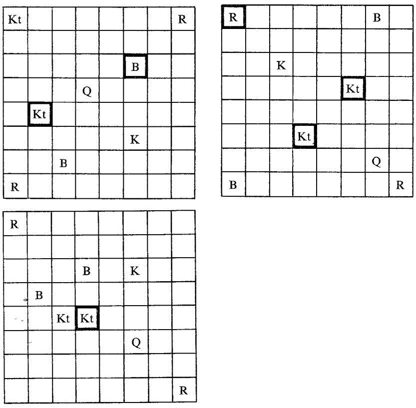
2—The problem:

Given the eight rear-rank pieces, place them on a board in such a way that they cover every square (i.e., any piece of the opposing color placed anywhere on the board may be taken in one move). The two bishops may not be of the same color.

Zagier again:

I have found several solutions. The problem as stated does not require one to cover one's own pieces (since they are not enemy pieces). Three solutions are shown at the top of the next column.

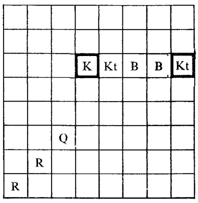
Darkened squares represent unprotected pieces; the third is the "best" in that all your own pieces but one are covered.



Kalman Shure, '51, writes:

The introduction of Puzzle Corner as a regular feature in Technology Review arouses mixed emotions within me. Puzzles tend to become like peanuts or pretzels with me—that is, I can't leave them alone; and although enjoying both, I can certainly do without them.

His solution to problem 2 is unique in that one of the bishops is superfluous.



He adds, "It should be evident that many other arrangements using the three ell-shaped and one square-shaped blocks can be made."

The problem was also solved by Toby Eisenstein, wife of Bruce A. Eisenstein, '63, Stephen A. Kliment, '53, Thomas L. De Fazio, '61, and George N. Krebs, Jr., '62.

If anyone can find a solution in which all the pieces are covered, I will welcome it.

3—Simplify the following:

$$\int_1^{\lim\limits_{z\to\infty}\left(1+\frac{1}{Z}\right)^z} \left(\frac{\pi}{2} - \tan^{-1}x + \frac{\sum\limits_{k=1}^{\infty}(-1)^{k+1}}{(2k+1)x^{2k+1}}\right)dx$$

$$-\frac{1}{2}(e^{i\alpha} - e^{-i\alpha})^2 + \cos 2\alpha$$

$$-\sum_{n=o}^{\infty} \frac{\cosh y\sqrt{1-\tanh^2 y}}{\left(\sum\limits_{j=o}^{\infty}\frac{\cosh\gamma\sqrt{1-\tanh^2\gamma}}{2^j}\right)^n}$$

The following solution was contributed by Mark B. Pelcovits, '70:

The upper limit of the integral is

$$r = \lim_{z\to\infty}\left(1+\frac{1}{Z}\right)^z.$$

In r = lim z ln (1 + 1/z) = lim (-1/(z + 1) + ln (1 + 1/z)) = 1 by l'Hôpital's rule. ∴ r = e.

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Puzzle Corner

(Continued from page 11)

$$\int_1^e \left(\frac{\pi}{2} - \tan^{-1} x + \frac{\sum_{k=1}^{\infty} (-1)^{k+1}}{(2k+1)\, x^{2k+1}} \right) dx$$

$$= \int_1^e \left(\frac{\pi}{2} - \frac{\pi}{2} + \frac{1}{x} - \frac{\sum_{k=1}^{\infty} (-1)^{k+1}}{(2k+1)\, x^{2k+1}} + \frac{\sum_{k=1}^{\infty} (-1)^{k+1}}{(2k+1) \times^{2k+1}} \right) dx$$

$$= \int_1^e \frac{dx}{x} = \ln x \Big]_1^e$$

$$= 1 - \tfrac{1}{2} (e^{i\alpha} - e^{-i\alpha})^2$$

$$= 2 \left(\frac{1}{4i^2} (e^{i\alpha} - e^{-i\alpha})^2 \right)$$

$$= 2 \sin^2\alpha = 1 - \cos 2\alpha$$

$$- \sum_{n=0}^{\infty} \left(\frac{\cosh y \sqrt{1 - \tanh^2 y}}{\sum_{j=0}^{\infty} \frac{\cosh \gamma \sqrt{1 - \tanh^2 \gamma}}{2^j}} \right)^n$$

$$= - \sum_{n=0}^{\infty} \left(\frac{1}{\sum_{j=0}^{\infty} \frac{1}{2^j}} \right)^n$$

(from the identity $\cosh y = \frac{1}{\sqrt{1 - \tanh^2 y}}$)

$= - \Sigma \frac{1}{2^n} = -2$ (from the sum of a geometric series)

$\therefore$ the expression equals $1 + 1 - \cos 2\alpha + \cos 2\alpha - 2 = 0$

Messrs. Zagier and Shure solved this problem as well.

4—What nonzero five-digit number has its digits reversed when multiplied by 4?

The following was received from Lawrence N. Smith, '68, who says the answer was arrived at by "sheer randomness":

$$\begin{array}{r} 21978 \\ 4 \\ \hline 87912 \end{array}$$

I received the following card from Thomas C. Lawford, Jr., '61, whose accuracy at solving puzzles is rivaled only by his fine judgment:

Answer to your puzzle problem 4 is 21978. (21978) (4)=87912. Appreciate your puzzle page, looks like a good idea to me. Best luck.

The following letter brought me particular joy:

I believe I have the solution to number 4: Would you believe 21978? It appears to me to be correct, but I might have made an error in my calculations—per usual.

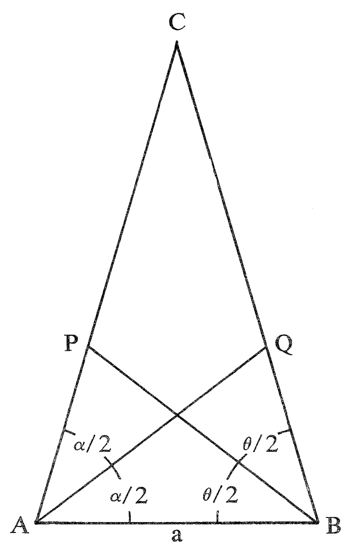
I would have solved a more difficult question but did not feel I should, in case I might insult the intelligence of an M.I.T. student. (Meant only sarcastically, I assure you. For I can only profess to be of the Class of '67 at Wellesley High School.)—Patricia Severance.

It is quite an honor for a grubby Tech tool to receive a letter from a senior at Wellesley—would you believe Wellesley High School? Thank you, Pat.

This problem was also answered by Messrs. Yu, Bator, Shure, Krebs, Joseph and Parker and by Zagier again, who even gave a uniqueness proof!

5—Prove the well-known theorem in geometry that if two angle bisectors of a triangle are equal, then the triangle is isosceles.

As usual, Zagier solved the problem. My proof has the advantage of being purely geometrical but his is so much hairier I cannot resist printing it instead.



We show that if one of the bisected angles is α, the other is also. Let AB be of length a, $\angle CAB = \alpha$. Let angle QBA be θ, so that AQ bisects A, BP bisects B, AQ=Y, PB=X. We have to show that X=Y implies $\theta = \alpha$. The sine law in ABP gives

$$\frac{x}{\sin\alpha} = \frac{a}{\sin(\pi - \alpha - \theta/2)} \text{ or}$$

$$x = \frac{a \sin \alpha}{\sin(\alpha + \theta/2)},$$

and similarly from ABQ,

$$Y = \frac{a \sin \theta}{\sin(\theta + \alpha/2)}.$$

Hence, $\frac{d}{d\theta}(x - y)$

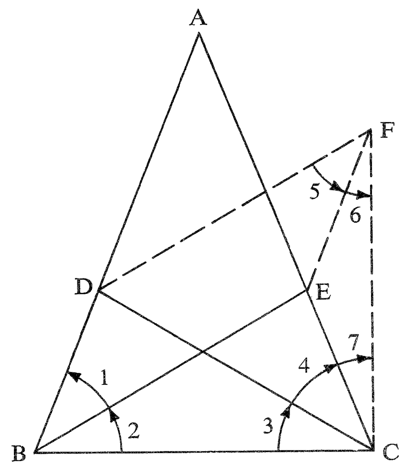
$$= - \frac{a \sin\alpha \cos(\alpha + \theta/2)}{2 \sin^2(\alpha + \theta/2)} - \frac{a \cos\theta \sin(\theta + \alpha/2) - a \sin\theta \cos(\theta + \alpha/2)}{\sin^2(\theta + \alpha/2)}$$

$$= - \frac{a \sin\alpha \cos(\alpha + \theta/2)}{2 \sin^2(\alpha + \theta/2} - a \frac{\sin \alpha/2}{\sin^2(\theta + \alpha/2)} < 0 \text{ for}$$

all θ, α, $a > 0$

That is, x-y is a monotone (decreasing) function of θ. Since plainly x-y=o when $\theta = \alpha$, this shows that $\theta = \alpha$ is the only root of x-y=o, so that the equality of the angle bisectors *x* and *y* implies that of the angles A and B, so ABC is isosceles.

Reductio ad absurdum proofs were submitted by Messrs. Shure and Joseph. My nickel came up heads, so here is Mr. Shure's version:



Given BE=CD, angle 1=angle 2, angle 3=angle 4.

Prove AB=AC or angle 1=angle 3= angle 2=angle 4.

Construct DF || BE and DF=BE, making BDFE a parallelogram.

Angle 1=angle 5: opposite angles of a parallelogram are equal.

DF=BE=CD: by construction and given.

Angle 5+angle 6=angle 4+angle 7: base angles of an isosceles triangle are equal.

If angle 1 > angle 4, then angle 5 > angle 4: angle 1=angle 5.

Therefore angle 7 > angle 6: angle 5 + angle 6=angle 4+angle 7.

In triangle EFC, EF > CE, since angle 7 > angle 6, and EF=BD, since opposite sides of a parallelogram are equal.

$\because$ BD > CE: EF > CE.

In triangles BCD and BCE, BC=BC (identity) and BE=CD (given).

Angle 3 > angle 2 (and therefore angle 4 > angle 1), since BD > CE.

But this is an impossibility, since we assumed angle 1 > angle 4.

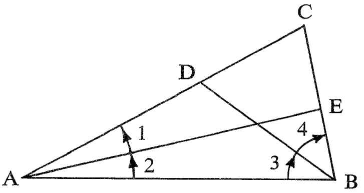
Therefore angle 1=angle 4=angle 3=angle 2 or BD=CE since it can also be shown that the assumption of angle 4 > angle 1 will also lead to an impossibility. Best wishes for the success of your column.

I received the following from Bradley C. Ross, '61:

Problem 5 in the November Technology Review is of interest for the tenacity with which it defies short, simple approaches to a solution despite its apparent simplicity. There is, however, a rather obscure theorem which facilitates a quick proof.

Problem: show that a triangle in which the bisectors of the base angles are equal in length is isosceles.

Given: triangle ABC in which angle 1=angle 2, angle 3=angle 4, and $\overline{AE}=\overline{BD}$.



Theorem: the square of an angle bisector is equal to the difference of the products of the adjacent sides and the segments of the opposite side.

$\overline{AC}\cdot\overline{AB}-\overline{CE}\cdot\overline{BE}=\overline{AE}^2=\overline{BD}^2=\overline{AB}\cdot\overline{BC}-\overline{AD}\cdot\overline{CD}$

Also

$\overline{AB}\cdot\overline{CE}=\overline{AC}\cdot\overline{BE}$ (1)

and

$\overline{AB}\cdot\overline{CD}=\overline{BC}\cdot\overline{AD}$, since angle bisectors divide the opposite side in proportion to the adjacent sides.

Thus,

$(\overline{AD}+\overline{CD})\cdot\overline{AB}+\overline{AD}\cdot\overline{CD}=\overline{AB}\cdot(\overline{BE}+\overline{CE})+\overline{CE}\cdot\overline{BE}$

$\overline{AD}\cdot(\overline{AB}+\overline{CD})+\overline{AB}\cdot\overline{CD}=\overline{BE}\cdot(\overline{AB}+\overline{CE})+\overline{AB}\cdot\overline{CE}$ using (1)

$\overline{AD}\cdot(\overline{AB}+\overline{CD}+\overline{CE})=\overline{BE}\cdot(\overline{AB}+\overline{CD}+\overline{CE})$

∴ $\overline{AD}=\overline{BE}$,

∴ Triangle AEB is congruent to triangle ABD,

∴ Angle 2=angle 3 and angle 1+ angle 2 =angle 3+angle 4,

∴ Triangle ABC is isosceles.

The above theorem is not too well known and perhaps deserves a proof.

It does, indeed, and I'll assign it as a problem in the next issue.

In his letter, Yu states, "Concerning #5, I know some guys who'll pay you plenty for its correct proof." The check may be made payable to Allan J. Gottlieb.