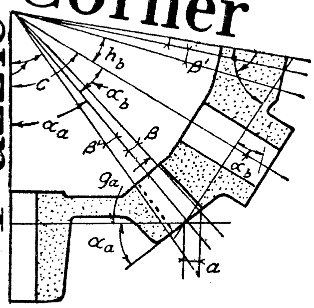
Puzzle Corner



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problems

15. If any two rays, emanating from the origin, are removed from the plane, the plane is cut into two components. What analogous statement(s) can be made with respect to the removal from three dimensional space of closed half-planes whose boundary lines pass through the origin? *(Thank you, MAA)*

16. This is another *Friedman Problem* for which, as yet, I have no solution. It is, in fact, a generalization of his first one (Problem 12, February). By the way, there was a slight mistake in February's Friedman problem. It should read, *the sum of finitely many elements.* As it was the problem was trivial, and we apologize to Mr. Friedman for such a blow to his image.
Let $C^n(k1,\ldots,km)$ be the set of all real numbers which can be represented in base n with digits only k1,...,km. For what values of n and m and for what m-tuples k1,....,km are all real numbers expressable as a *(finite)* sum of elements of $C^n(k1,\ldots,km)$?

17. Now for some easier problems. The solution to this problem will help in solving many others.
Find criteria for divisibility by 3,9,11.

18. OK, you chess hacks, here's one for you.
Given: an 8 x 8 chessboard and a knight which *you* may place at will.
Find: a sequence of 63 consecutive moves such that after having completed this tour the knight has visited each square.

19. People keep telling me that computer hacking (oops — I mean programming) is mathematics. Without taking a stand on the issue, I propose the following:
You choose the programming language. Write a program which will read a card, remove all vowels (abcdefg becomes bcdfg — *no* spaces added), punch the resulting card (all extra spaces will be on the right), and go back and read another card. What I will do is to count the total number of characters used in your solution. Lowest number wins. (Yes, I will accept computer problems for publication.)

SPEED *Department*

20. Prove that U must be zero.

$$\begin{array}{r} \text{IS} \\ \underline{\text{GAP}} \\ \text{UAP?} \end{array}$$

The operation is addition.

(My deadline was February 20)

21. If abc is prime, how many prime factors has abcabc?

answers

Due to space considerations, the questions are not reprinted when giving solutions.

1. Consider the following triangle:

$$\begin{array}{cccc} \frac{1}{C_0{}^0} & \frac{1}{2C_1{}^1} & \frac{1}{3C_2{}^2} & \cdots \\ & \frac{1}{2C_1{}^0} & \frac{1}{3C_2{}^1} & \cdots \\ & & \frac{1}{3C_2{}^0} & \cdots \end{array}$$

LEMMA: This triangle is the difference triangle for the sequence $a_n = 1/n$ (modulo minus signs, which should appear in the even numbered rows). To prove the lemma it will suffice to show that for any $k \leq n\text{-}1$, the following is true:

$$\frac{1}{nC_{n-1}{}^k} - \frac{1}{(n+1)C_n{}^{k+1}} = \frac{1}{(n+1)C_n{}^k}$$

but note that:

$$\frac{1}{nC_{n-1}{}^k} - \frac{1}{(n+1)C_n{}^{k+1}}$$

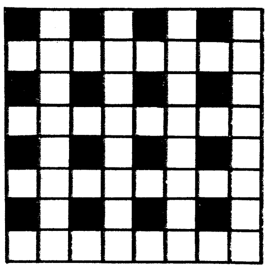
$$= \frac{k!}{n(n-1)(n-2)\ldots(n-k)} - \frac{(k+1)!}{(n+1)n(n-1)\ldots(n-k)}$$

$$= \frac{k!}{n(n-1)\ldots(n-k)}\left(1 - \frac{k+1}{n+1}\right)$$

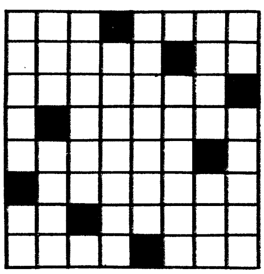
$$= \frac{k!}{n(n-1)\ldots(n-k)}\left(\frac{n-k}{n+1}\right) = \frac{1}{(n+1)C_n{}^k}$$

QED LEMMA The verification that the operations of the problem when applied to the above triangle yield a Pascal Triangle is immediate.

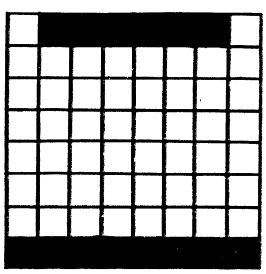
2. Bill Friedmann, '66, solved this one. Unfortunately his answer for part (d) is incorrect. The following are his answers (part (d) revised):



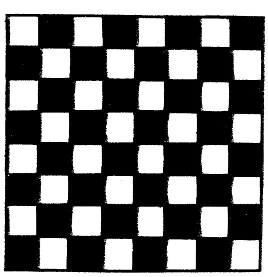
Kings — 16



Queens — 8
(this can be used
for rooks as well)



Bishops — 14



Knights — 32

3. This problem has led to some rather interesting activity. Ed Fiala, '66, who submitted the problem and gave it to the pledges (did he ever), offers the following solution:

The first step is to figure out the result of the various opposing strategies; write a matrix showing the Tech Men's expected winnings:

	3-0	2-1	1-2	0-3
4-0	4	2	1	0
3-1	1	3	0	-1
2-2	-2	2	2	-2
1-3	-1	0	3	1
0-4	0	1	2	4

Take advantage of the symmetry of the problem — 4-0 and 0-4 should be played the same fraction of the time, for example. The matrix then becomes:

	0-3 3-0	1-2 2-1
4-0 0-4	2	3/2
3-1 1-3	0	3/2
2-2	-2	2

Since the (4-0 or 0-4) strategy gives an expected gain greater than or equal to the (3-1 or 1-3) strategy, the Tech Men never play 3-1 or 1-3. Now rewrite the matrix and assign probabilities to each strategy:

	Y 0-3 3-0	1-Y 1-2 2-1
4-0 0-4	2	3/2
2-2	-2	2

The expectation E is $2xy + (3/2)x(1-y) - 2(1-x)y + 2(1-x)(1-y)$. For Tech Men to maximize over their variable, $\partial E/\partial x$ must be zero, and for Harvies to minimize over their variable, $\partial E/\partial y$ must be zero.

But now the plot thickens. Enter Samuel Wagstaff, Jr., '66 (and his PDP-1), with the following letter:

Dear Allan,

Here is the solution to the third problem in the TEN Puzzle Corner. Since the problem was given by an EE man to EE people, I thought it was fair to use EE facilities to find the solution. I have written a program for the EE PDP-1 computer which solves all two person games with pay-off matrices of size 10 x 10 or smaller. (I did not write the program specifically for this problem; it can be solved in one minute by dominance methods. I wrote the program about a month ago for 18.23 Game Theory Seminar.) The program considers all square submatrices of the original matrix, finds the all-strategies-active solution of the submatrix, and checks to see if that solution is a solution of the original matrix. If it is, the solution is typed out. The numbers following "Row strategy" are the row oddments, i.e. the proportion of the time each row should be played. Ditto for the columns. To get the true strategy vectors, divide those given by the denominator of the following value: Note that the value is 5/2 in all cases (which is nice). The program does not compare the solutions that it types out; therefore, in games which have more than one solution, as this one, there is some redundancy in the solutions it types out. (In this game there were 3 x 3 submatrices which had legal solutions. These appear as 2 x 2 solutions because one row strategy and one column strategy were zero.) The typewriter missed a carriage return while printing the fourth solution; this was not my fault. The running time for a 5 x 4 matrix is about 2 or 3 seconds, not counting the typeout period. A 7 x 7 matrix takes about a minute. Matrices larger than 8 x 8 have running times that makes their solution by computer impractical, but I have written another program which types out a series of approximations to the solution as fast as the typewriter will go. This program will handle matrices of arbitrary size.

The results are to be interpreted as follows: The MIT students should choose the first and last row with equal probability, i.e. they send all 4 men to either the Wellesley or Smith girl, and decide which one by flipping a coin. The Harvies can do just about anything, just so long as they don't let MIT know what they have decided to do. On the average, the MIT men will get 5/2 points. This means (in terms of the problem, not just the matrix) that MIT will certainly get the girl they choose, and on the average they will kill one and a half Harvies. If I have interpreted the problem wrongly, and have the wrong matrix, please send me the correct matrix, and I will run it. If you have any other matrices (smaller than 9 x 9) that you want solved, send them to me, and I will return the solutions promptly. By the way, (792)(8907) = 7054344.

Sincerely yours,
Samuel Wagstaff, Jr.
Wood 205, East Campus

To add to the confusion, Bill adds the following:

	4-0	3-1	2-2	1-3	0-4
3-0	4	1	1	1	1
2-1	3	3	2	2	2
1-2	2	2	2	3	3
0-3	1	1	1	1	4

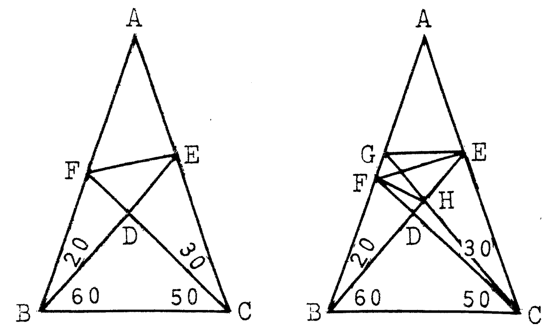
the payoffs are for MIT

He proposes the following strategies:

MIT — 4-0 and 0-4 dominate and hence should be played 50% each.
Harvard — 3-0 and 0-3 should be played x% each, x e 50; 2-1 and 1-2 should be played (50 - x)% each.

This solution is very similar to the one submitted by the Wagstaff — PDP-1 team. Rather than arbitrate this dispute, I will let the issue dangle. If any of the combatants has a rebuttal to give he should send it to me and I will be glad to print all that I receive, within reason.

4. Bill tells me that he believes that the answer is 30° . Theodore C.-C. Chang (TC^3), who proposed this problem, informs me that Bill is correct, and that it is quite reasonable that Bill remembers the answer but not the proof. You see, they were on the same high school math team when the problem was first brought up. The following solution was submitted by TC^3:



FIND X.
We draw line EG parallel to line BC, and line GC
H is the intersection of EB and GC
connect FH
angle BFC is 50°
so BFC is isosceles, and BF equals BC
and HBC is equilateral, and BC equals BH
so angle BHF is 80°
angle FHE is 100°
note that angle FGE is 180° - AGE equals 100
also note that EGH is equilateral
so GE equals EH
angle HGF equals 180° - 80° - 60° equals 40°
angle FHG equals FHE - 60° equals 40°
therefore, FH equals FG
By SAS, we have triangle FHE congruent to triangle FGE
so x is 1/2 of angle HEG equals 30°

5. Once again the solution that I am printing is due to Bill — Congratulations, I wish I had a dozen more like you. By the way, Sam Wagstaff also solved the problem. Bill's solution is as follows:

Let L be 70xy34z.
Since 792=(8)(9)(11), L is divisible by 8.
Therefore, 34z is divisible by 8 (i.e., z = 4 and L = 70xy344).
L divisible by 9 implies that 7 + x + y + 3 + 4 + 4 is divisible by 9.
Thus x + y = 9.
L divisible by 11 implies that 7 + x + 3 + 4 − y − 4 is divisible by 110.
Thus, x and y are determined and L is 7054344. ■