It is easy to see that the minimum number of posts required is 4. This is because any single light post will fail to illuminate some great circle. (Imagine that the light post is at the north pole and the great circle is the equator.) From that point, it is not possible to illuminate that great circle with less than three additional posts.

The simplest solution, and almost certainly the one with the shortest total post length, comes from placing the posts at the corners of an inscribed regular tetrahedron as illustrated in Figure 1. This gives us a total lamppost length of 8*R* where *R* is the radius of the planet. The trick is to prove that this is the minimum length. To do this, we make one clever observation and then use brute force.

Any minimal lighting solution must correspond to the case of a (not necessarily regular) tetrahedron inscribed in the circle (this is not the inscribed tetrahedron described in the last paragraph). This is because the light from a single post creates a circular patch on the sphere that must contain the "triangular" region given by the space not covered by the other posts. The minimum post height comes from minimizing the radius of the circle to the one that contains the three corners of the triangles. The light pattern appears as in Figure 2 below.

We are working in the case of four points at random on a sphere given by $x^2 + y^2 + z^2 = 1$. For any face of the tetrahedron, we can find a circumscribed circle. The lamppost height corresponding to that circle is given by similar triangles as illustrated in Figure 3. We see that the height of the can be found with the help of the ratio

$$\frac{\sqrt{R^2 - r^2}}{R} = \frac{R}{R+h}. \tag{1}$$

It will be given by the formula

$$h' = \left(1 - r'^2\right)^{-1/2} - 1 \tag{2}$$

where $h' = h/R$ and $r' = r/R$. At this point, we will just assume a sphere of unit radius. We make use of the past Puzzle Corner where it was shown that the radius of a circumscribed circle for a triangle with sides *a*, *b*, and *c* is given by

$$r = abc/(4K) \tag{3}$$

where *K* is the area of the triangle. If we use the vectors as given in Figure 4, then the formula for the areas of those triangles will be given by $\frac{1}{2}\|\vec{A}\times\vec{B}\|$ etc. This gives us a final formula of

$$\sum h = \left(1 - \frac{\|\vec{A}\|^2 \|\vec{B}\|^2 \|\vec{D}\|^2}{4\|\vec{A}\times\vec{B}\|^2}\right)^{-1/2} + \left(1 - \frac{\|\vec{A}\|^2 \|\vec{C}\|^2 \|\vec{E}\|^2}{4\|\vec{A}\times\vec{C}\|^2}\right)^{-1/2} + \left(1 - \frac{\|\vec{B}\|^2 \|\vec{C}\|^2 \|\vec{F}\|^2}{4\|\vec{B}\times\vec{C}\|^2}\right)^{-1/2} + \left(1 - \frac{\|\vec{D}\|^2 \|\vec{E}\|^2 \|\vec{F}\|^2}{4\|\vec{D}\times\vec{E}\|^2}\right)^{-1/2} - 4 \tag{4}$$

The brute force part comes from generating random inscribed tetrahedrons and determining the post length that they correspond to. We are solving the problem by throwing darts at it. We pick the points (without loss of generality, we may pick one point to be the north pole and one point to be on the x-axis) and check to make sure that the center of the sphere is contained in the tetrahedron. If it is, we measure the post heights. For instance, for 1,000,000 sets of throws, we get a minimum height of 8.0497001 and the number of values distributed (for each of 10 equal intervals) between 8 and 9: 3,9,21,32,47,62,77,88,90,117. This is a clear indication that the minimum is at 8 (well, clear enough for me).

Figure 1: The minimal lighting solution. The sphere is inscribed in a regular tetrahedron. The bases of the posts are the corners of an inscribed regular tetrahedron (not shown).

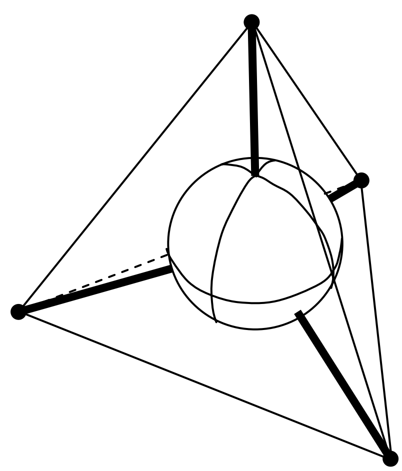

Figure 2: The unlit “triangle” and its circumscribed circle.

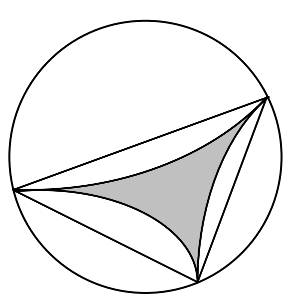

Figure 3: Similar triangles giving the relationship between the radius of a circle (on the surface of a sphere) and the minimal lamppost height required to illuminate it.

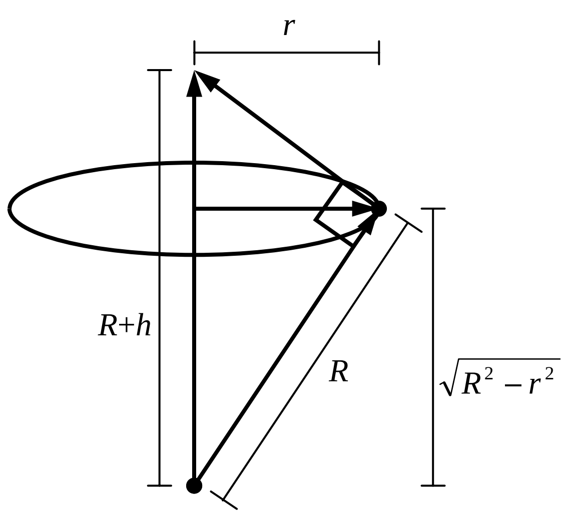

Figure 4: Illustration of the vectors used in Eqn. 4. The points P_i all lie on the surface of a sphere with radius R.