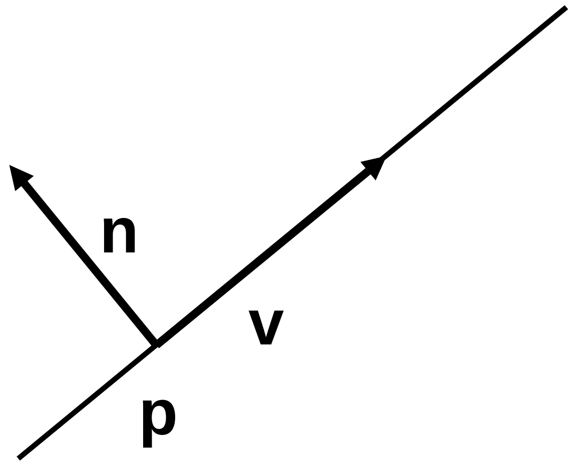
Geometry review, part II

Line equations



parametric equation:

$$\mathbf{q}(t) = \mathbf{p} + \mathbf{v}t$$

t **= parameter**

p = point on the line

n = unit normal

v = unit vector along

implicit equation:

$$(\mathbf{n}\cdot(\mathbf{q} - \mathbf{p})) = 0$$

if we know n = $[n_x, n_y]$, then we can take v to be v = $[n_y, -n_x]$, and the other way around.

Line equations

Intersecting two lines:

take one in implicit form: $((q - p^1) \cdot n^1) = 0$

the other in parametric: $q = p^2 + v^2 t$

If $q^i = p^2 + v^2 t^i$ **is the intersection point, it satisfies both equations.**

Plug parametric into implicit, solve for t^i **:**

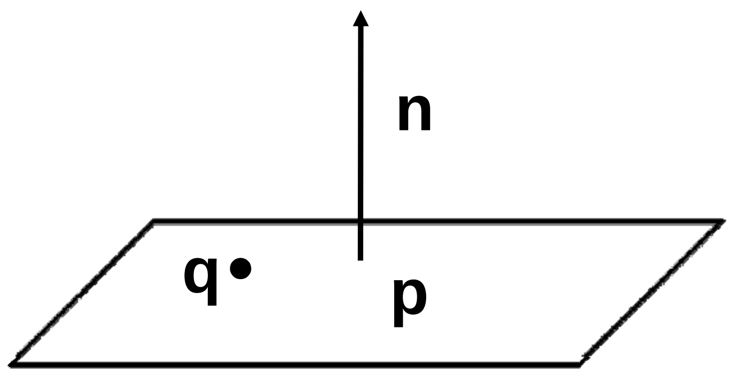
$$((p^2 + v^2 t^i - p^1) \cdot n^2) = 0$$

If $(v^2 \cdot n^1) \neq 0$**, then** $t^i = -\frac{(p^2 - p^1 \cdot n^1)}{(v^2 \cdot n^1)}$

Otherwise, the lines are parallel or coincide.

Plane equations

implicit equation: $(q-p) \cdot n = 0$, exactly like line in 2D!



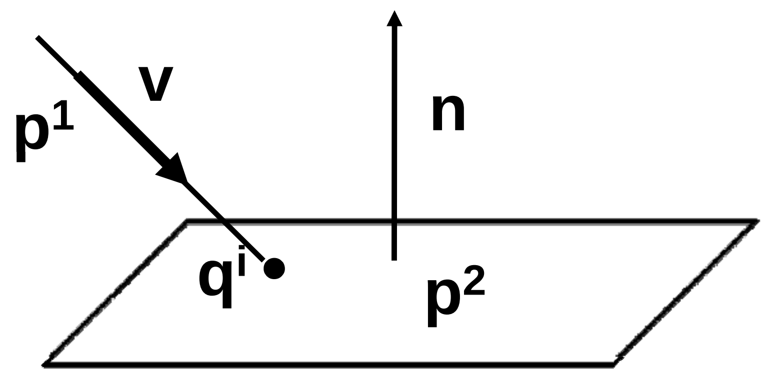
parametric equation: 2 parameters t_1, t_2

$q(t_1, t_2) = v_1 t_1 + v_2 t_2$, where v_1 and v_2 are two vectors in the plane.

$$\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{n}$$

Intersecting a line and a plane

Same old trick: use the parametric equation for the line, implicit for the plane.



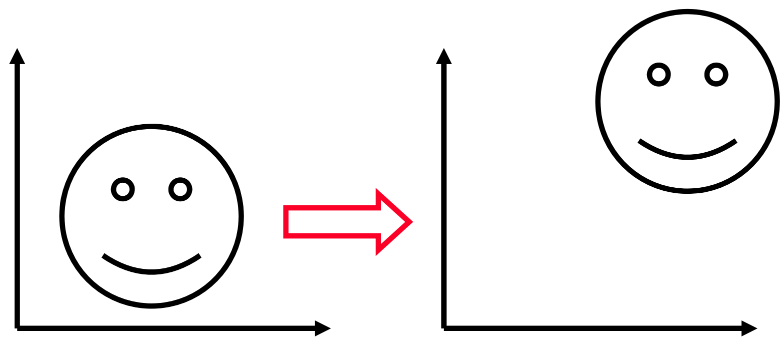
$$(\mathbf{p}^1 + \mathbf{v}t^i - \mathbf{p}^2 \cdot \mathbf{n}) = 0$$

$$t^i = -\frac{(\mathbf{p}^1 - \mathbf{p}^2 \cdot \mathbf{n})}{(\mathbf{v} \cdot \mathbf{n})}$$

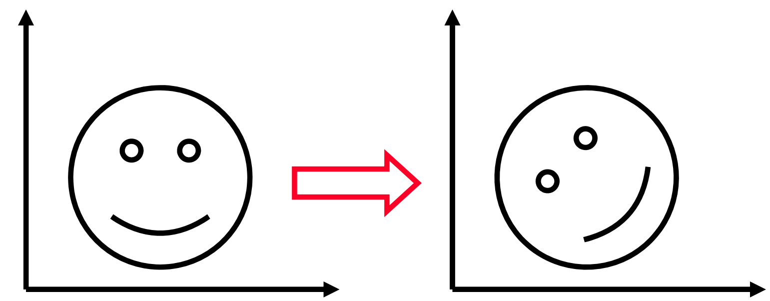
Do not forget to check for zero in the denominator!

Transformations

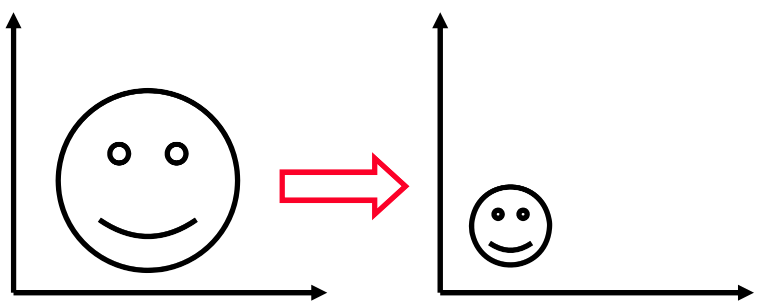
Examples of transformations:



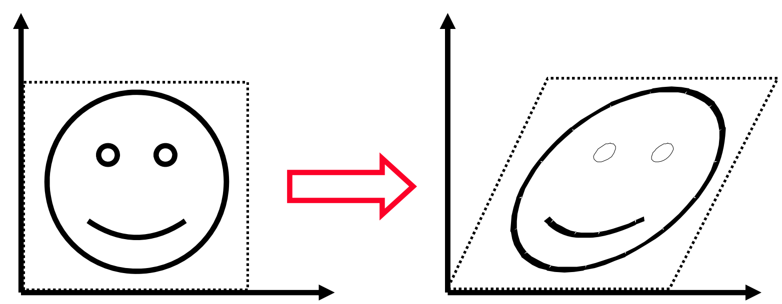
translation



rotation



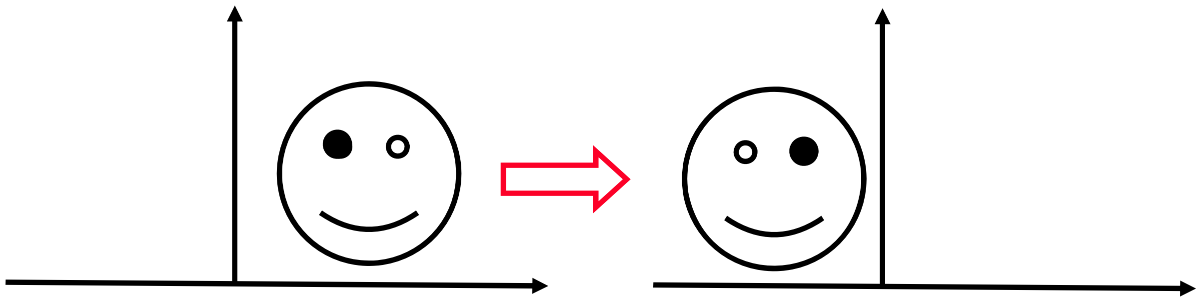
scaling



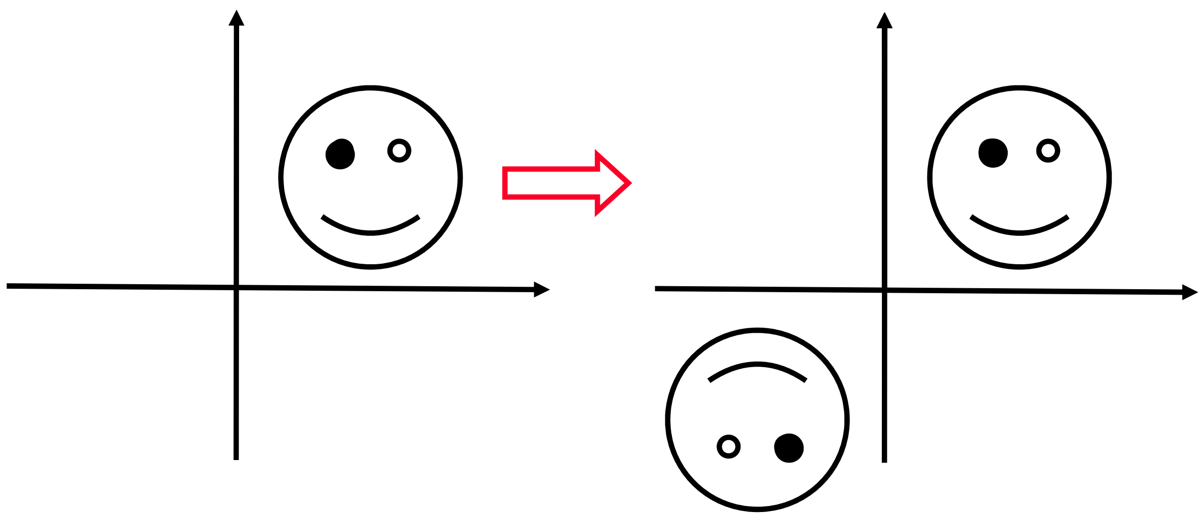
shear

Transformations

More examples:



reflection with respect to the y-axis



reflection with respect to the origin

Transformations

Linear transformations: take straight lines to straight lines.

All of the examples are linear.

Affine transformations: take paralllel lines to parallel lines.

All of the examples are affine,

an example of linear non-affine is perspective projection.

Orthogonal transformations: preserve distances, move all objects as rigid bodies.

rotation, translation and reflections are affine.

Composition of transformations

- Order matters! (rotation * translation ≠ translation * rotation)
- Composition of transformations = matrix multiplication:
  if T is a rotation and S is a scaling, then applying scaling first and rotation second is the same as applying transformation given by the matrix TS (note the order).
- Reversing the order does not work in most cases

Transformations and matrices

Any affine transformation can be written as

$$\begin{pmatrix} \mathbf{x'} \\ \mathbf{y'} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} + \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} \qquad \mathbf{p'=Ap}$$

Images of basis vectors under affine transformations:

$$\mathbf{e}_\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(column form of writing vectors)

$$\mathbf{Ae}_\mathbf{x} = \begin{pmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{11} \\ \mathbf{a}_{21} \end{pmatrix} \qquad \mathbf{Ae}_\mathbf{y} = \begin{pmatrix} \mathbf{a}_{21} \\ \mathbf{a}_{22} \end{pmatrix}$$

$$\mathbf{e}_\mathbf{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Transformations and matrices

Matrices of some transformations:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$ **shear**

$$\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$$ **scale by factor s**

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$ **rotation**

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$ **reflection with respect to the origin**

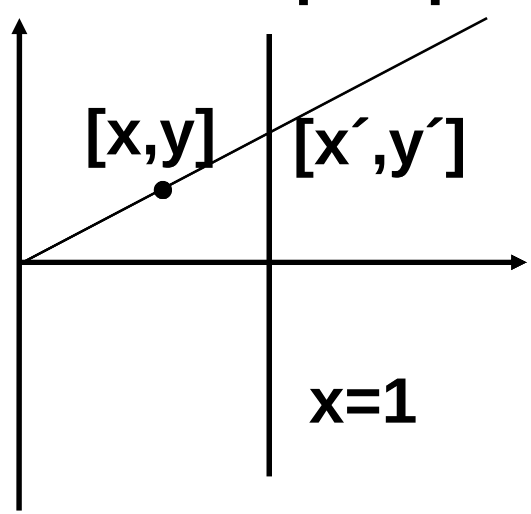
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$ **reflection with respect to y-axis**

Homogeneous coordinates

Problem

Even for affine transformations we cannot write them as a single 2x2 matrix; we need an additional vector for translations.

We cannot write all linear transformations even in the form Ax +b where A is a 2x2 matrix and b is a 2d vector. Example: perspective projection



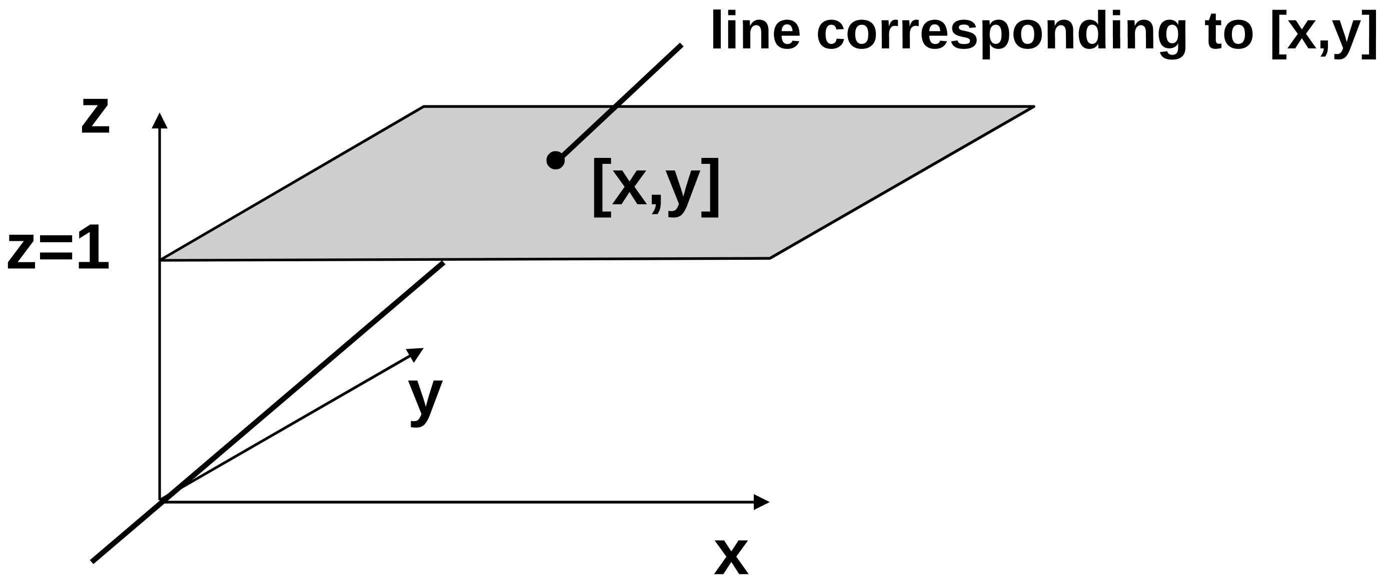
$$x' = 1$$
$$y' = y/x$$

equations not linear!

Homogeneous coordinates

- replace 2d points with 3d points, last coordinate 1
- for a 3d point (x,y,w) the corresponding 2d point is (x/w,y/w) if w is not zero
- each 2d point (x,y) corresponds to a line in 3d; all points on this line can be written as [kx,ky,k] for some k.
- (x,y,0) does not correspond to a 2d point, corresponds to a direction (will discuss later)
- Geometric construction: 3d points are mapped to 2d points by projection to the plane z =1 from the origin

Homogeneous coordinates



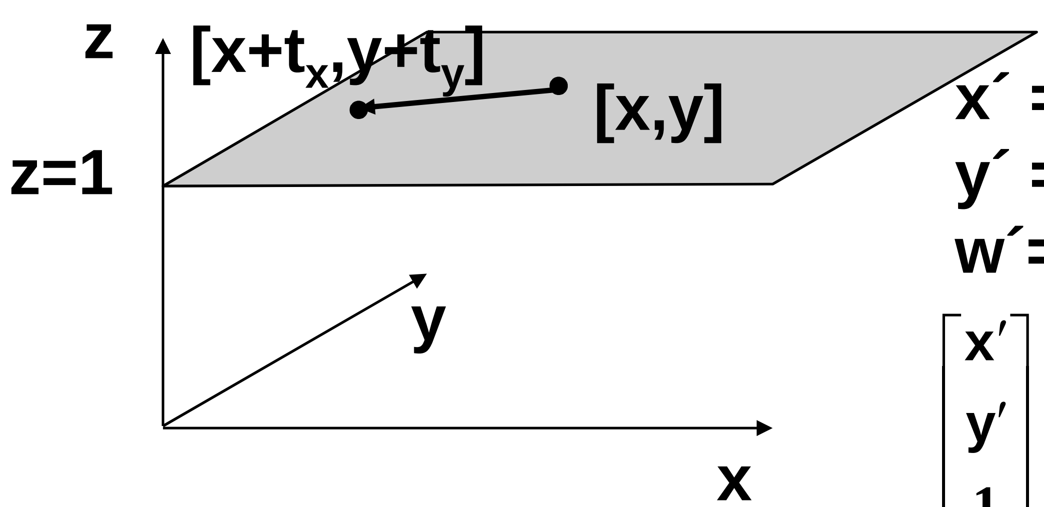
From homogeneous to 2d: [x,y,w] becomes [x/w,y/w]
From 2d to homogeneous: [x,y] becomes [kx,ky,k]
(can pick any nonzero k!)

Homogeneous transformations

Any linear transformation can be written in matrix form in homogeneous coordinates.

Example 1: translations

[x,y] in hom. coords is [x,y,1]



$x' = x+t_x = x + t_x \cdot 1$
$y' = y+t_y = y+t_y \cdot 1$
$w' = 1$

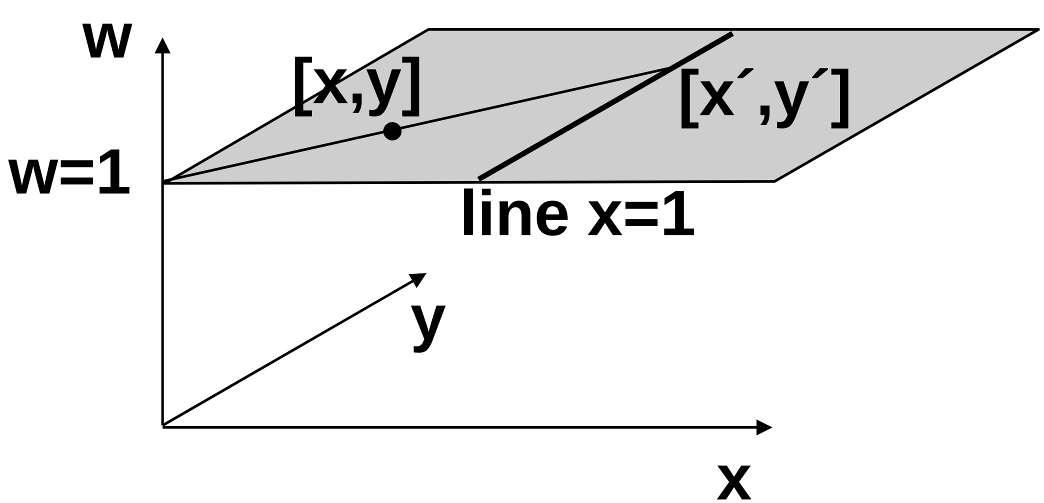
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous transformations

Example 2: perspective projection

$x' = 1$
$y' = y/x$
$w' = 1$

Can multiply all three components by the same number -- the 2D point won't change! Multiply by x.



$x' = x$
$y' = y$
$w' = x$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Matrices of basic transformations

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ rotation

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$ translation

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ scaling

$$\begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ skew

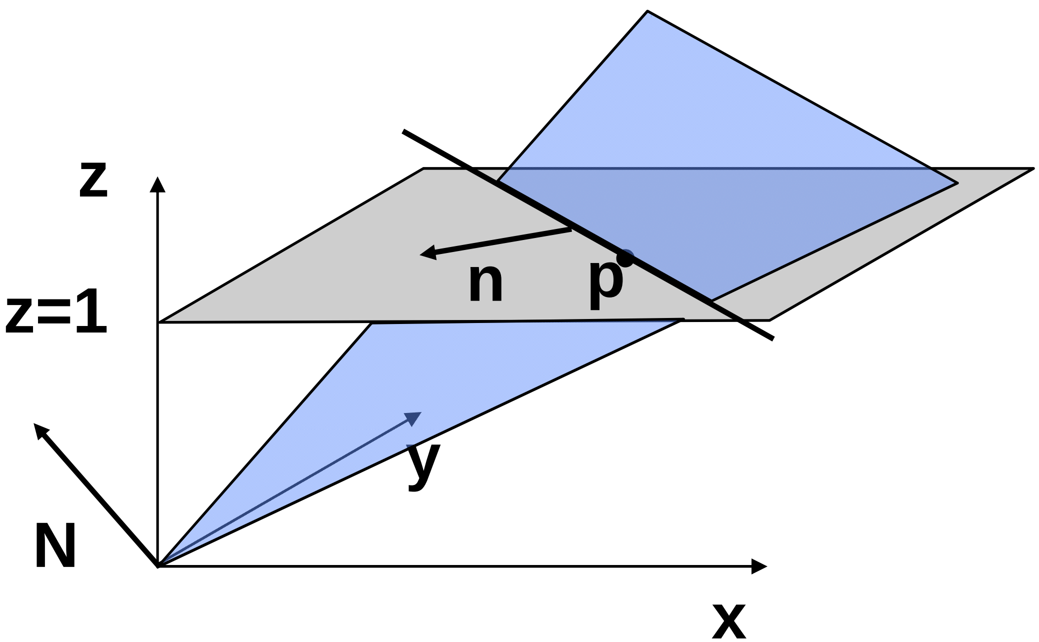
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$ general affine transform

Homogeneous line equation

Implicit line equation in 2D: (n · (q-p)) = 0,

n = 2D vector, p = 2D point on the line.

Goal: rewrite in homogeneous coordinates.



2D point corresponds to a 3D line through origin; 2D line corresponds to a plane through origin

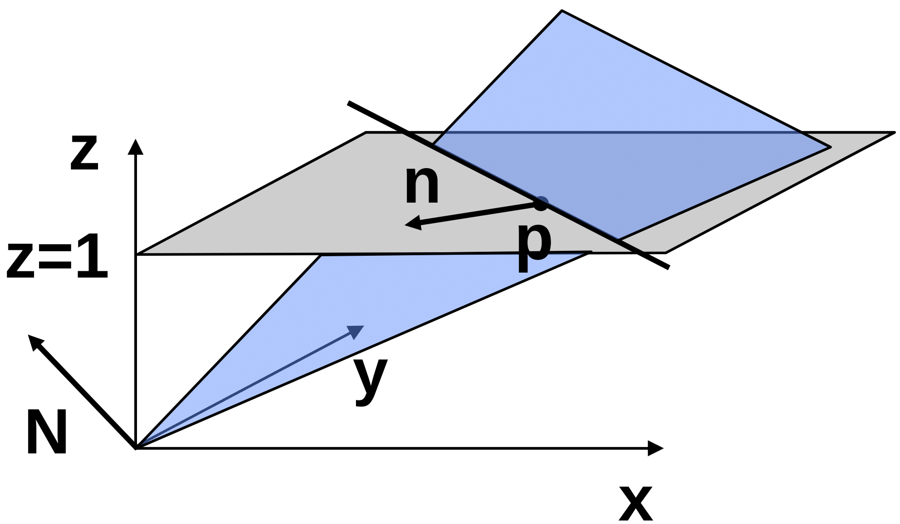
In other words, the 2D line is intersection of a plane through origin with the plane z=1.

Homogeneous line equation

Rewrite the line equation:

$$(n,q-p)=n_x x+n_y y+(n,-p)=([n_x,n_{y,}-(n,p)],[x,y,1])=(N,\overline{q})$$

where $N=[n_x,n_y,-(n,p)]$ is the normal to the plane corresponding to the line, and $\overline{q}$ is the homogeneous form of $q=[x,y]$: $\overline{q}=[x,y,1]$



Homogeneous form of the line equation:

$$(N\cdot\overline{q})=0$$

Homogeneous coordinates

regular 3D point to homogeneous:

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \longrightarrow \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}$$

homogeneous point to regular 3D:

$$\begin{pmatrix} p_x \\ p_y \\ p_z \\ p_w \end{pmatrix} \longrightarrow \begin{pmatrix} p_x/p_w \\ p_y/p_w \\ p_z/p_w \end{pmatrix}$$

Translation and scaling

Similar to 2D; translation by a vector

$$t = [t_x, t_y, t_z]$$

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Nonuniform scaling in three directions

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotations around coord axes

angle θ, around X axis:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

around Y axis:

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

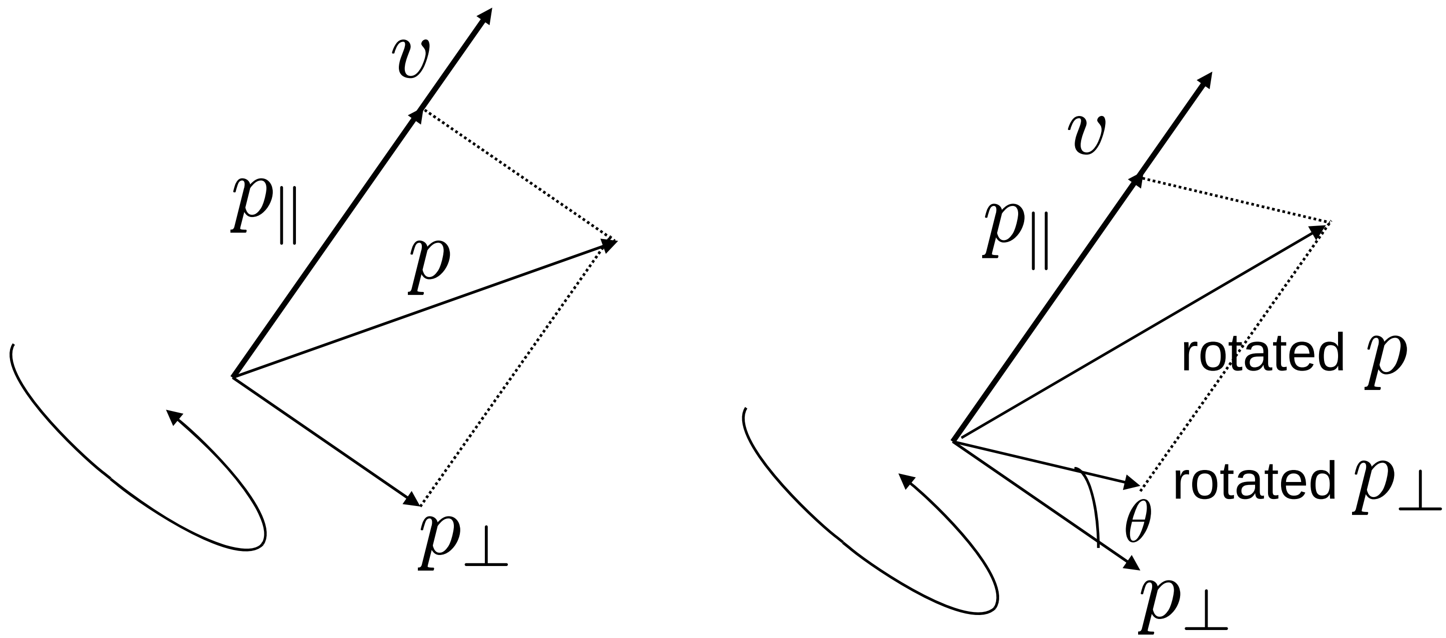
note where the minus is!

around Z axis:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

General rotations

Given an axis (a unit vector) and an angle, find the matrix



Only the component perpendicular to axis changes

General rotations

(rotated vectors are denoted with $'$)

project p on v: $$p_{||} = (p, v)v$$

the rest of p is the other component: $$p_{\perp} = p - (p, v)v$$

rotate perp. component: $$p'_{\perp} = p_{\perp} \cos\theta + (v \times p_{\perp}) \sin\theta$$

add back two components: $$p' = p'_{\perp} + p_{||}$$

Combine everything, using $v \times p_{\perp} = v \times p$ to simplify:

$$p' = \cos\theta \; p + (1 - \cos\theta)(p, v)v + \sin\theta(v \times p)$$

General rotations

How do we write all this using matrices?

$$p' = \cos\theta \; p + (1-\cos\theta)(p,v)v + \sin\theta(v \times p)$$

$$(p,v)v = \begin{bmatrix} v_x v_x p_x + v_x v_y p_y + v_x v_z p_z \\ v_y v_x p_x + v_y v_y p_y + v_y v_z p_z \\ v_z v_x p_x + v_z v_y p_y + v_z v_z p_z \end{bmatrix} = \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z \\ v_y v_x & v_y v_y & v_y v_z \\ v_z v_x & v_z v_y & v_z v_z \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$(v \times p) = \begin{bmatrix} -v_z p_y + v_y p_z \\ v_z p_x - v_x p_z \\ -v_y p_x + v_x p_y \end{bmatrix} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

Final result, the matrix for a general rotation around a by angle θ:

$$\cos\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1-\cos\theta) \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z \\ v_y v_x & v_y v_y & v_y v_z \\ v_z v_x & v_z v_y & v_z v_z \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}$$

Composition of transformations

- Order matters! (rotation * translation ≠ translation * rotation)
- Composition of transformations = matrix multiplication:
  if T is a rotation and S is a scaling, then applying scaling first and rotation second is the same as applying transformation given by the matrix TS (note the order).
- Reversing the order does not work in most cases

Transformation order

- **When we write transformations using standard math notation, the closest transformation to the point is applied first:**

$$T\ R\,S\,p = T(R(Sp))$$

- **first, the object is scaled,**
  then rotated,
  then translated
- **This is the most common transformation order for an object (scale-rotate-translate)**