Geometry review, part I

Geometry review I

Vectors and points

- points and vectors
- Geometric vs. coordinate-based (algebraic) approach
- operations on vectors and points

Lines

- implicit and parametric equations
- intersections, parallel lines

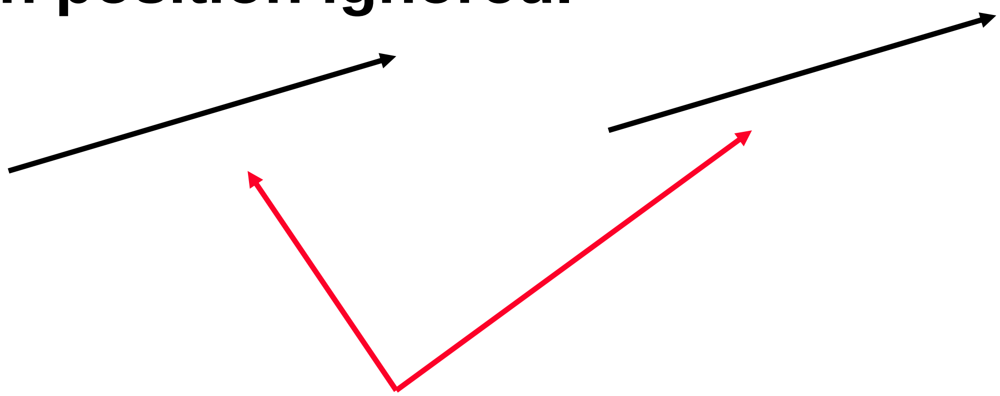
Planes

- implicit and parametric equations
- intersections with lines

Geometry vs. coordinates

<u>Geometric view:</u> a vector is a directed line segment,

with position ignored.



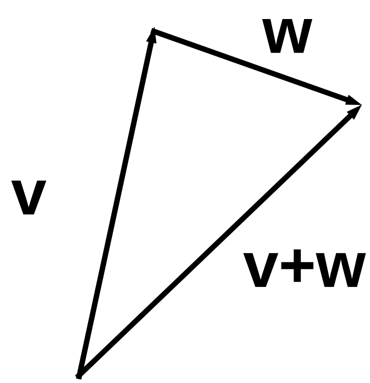
different line segments (but with the same length and direction) define the same vector

A vector can be thought of as a translation.

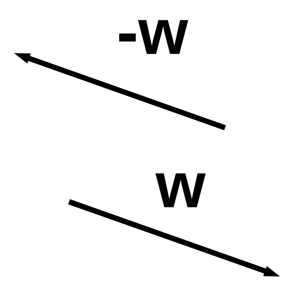
<u>Algebraic view:</u> a vector is a pair of numbers

Vectors and points

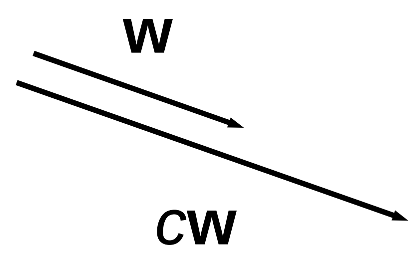
Vector = directed segment with position ignored



addition



negation



multiplication by number

Operations on points and vectors:

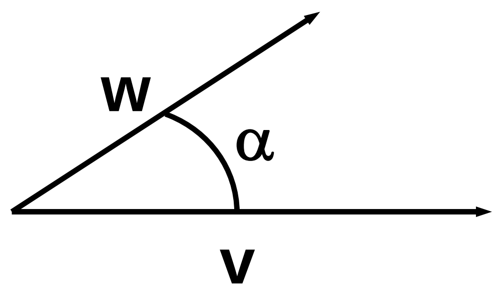
point - point = vector

point + vector = point

Dot product

Dot product: used to compute projections, angles and lengths.

Notation: $(w \cdot v)$ = dot product of vectors w and v.



$(w \cdot v) = |w|\,|v| \cos\alpha$,

$|v|$ = length of v

Properties:

- if w and v are perpendicular, $(w \cdot v) = 0$
- $(w \cdot w) = |w|^2$
- angle between w and v: $\cos\alpha = (w \cdot v)/|w||v|$
- length of projection of w on v: $(w \cdot v)/|v|$

Coordinate systems

For computations, vectors can be described as pairs (2D), triples (3D), … of numbers.

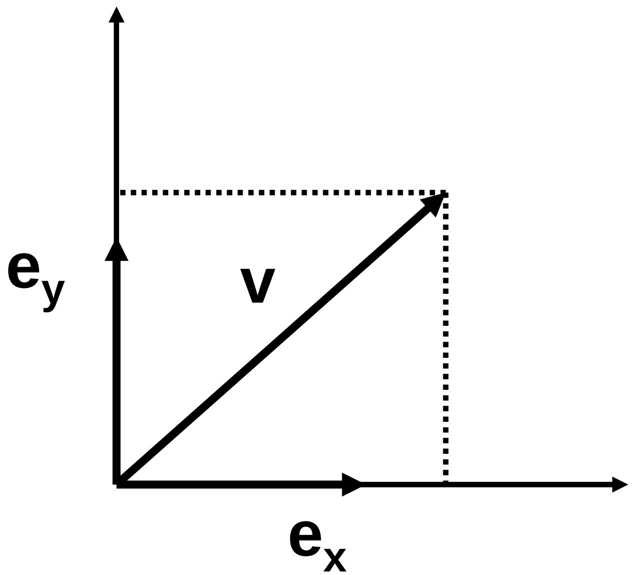
Coordinate system (2D) = point (origin) + 2 basis vectors.

Orthogonal coordinate system: basis vectors perpendicular.

Orthonormal coordinate system: basis vectors perpendicular and of unit length.

Representation of a vector in a coordinate system: 2 numbers equal to the lengths (signed) of projections on basis vectors.

Operations in coordinates



$$v = \underbrace{(v\cdot e_x)}_{v_x} e_x + \underbrace{(v\cdot e_y)}_{v_y} e_y$$

works only for orthonormal coordinates!

$$v = v_x\, e_x + v_y\, e_y = [\, v_x, v_y \,]$$

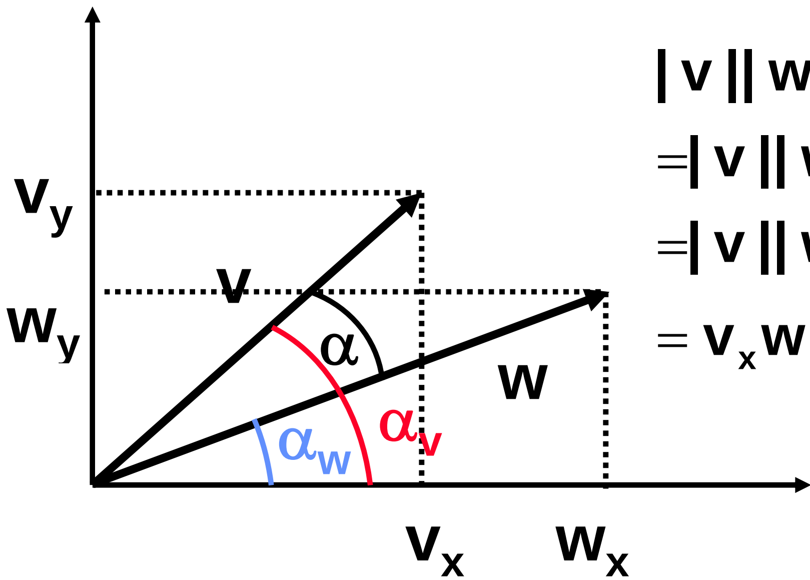
Operations in coordinate form:

$v + w = [\, v_x, v_y \,] + [\, w_x, w_y \,] = [v_x + w_x, v_y + w_y]$

$-w = [\, -w_x, -w_y \,]$

$\alpha\, w = = [\alpha\, w_x, \alpha\, w_y \,]$

Dot product in coordinates



$$|\mathbf{v}||\mathbf{w}|\cos\alpha = |\mathbf{v}||\mathbf{w}|\cos(\alpha_v - \alpha_w)$$
$$= |\mathbf{v}||\mathbf{w}|(\cos\alpha_v \cos\alpha_w + \sin\alpha_v \sin\alpha_w)$$
$$= |\mathbf{v}||\mathbf{w}|(v_x w_x / |\mathbf{v}||\mathbf{w}| + v_y w_y / |\mathbf{v}||\mathbf{w}|)$$
$$= v_x w_x + v_y w_y$$

Linear properties become obvious:

((v+w)·u) = (v·u) + (w·u)

(*a*v·w) = *a*(v·w)

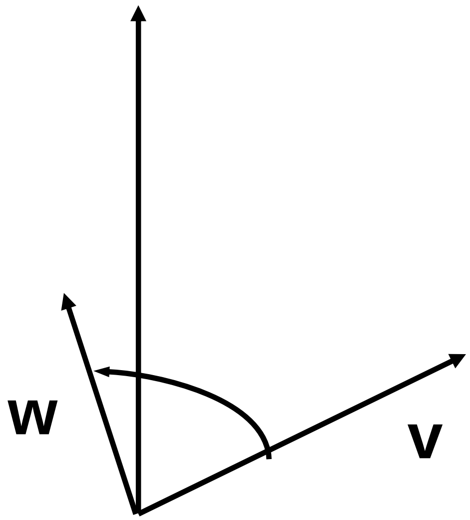
3D vectors

Same as 2D (directed line segments with position ignored), but we have different properties.

- In 2D, the vector perpendicular to a given vector is unique (up to a scale).
- In 3D, it is not.
- Two 3D vectors in 3D can be multiplied to get a vector (vector or cross product).
- Dot product works the same way, but the coordinate expression is

$$(v \cdot w) = v_x w_x + v_y w_y + v_z w_z$$

Vector (cross) product



$\mathbf{v} \times \mathbf{w}$ **has length** $|\mathbf{v}||\mathbf{w}|\,\mathbf{sin}\,\alpha$

= area of the parallelogram with two sides given by v and w, and is perpendicular to the plane of v and w.

$$(\mathbf{v}+\mathbf{w})\times\mathbf{u} = \mathbf{v}\times\mathbf{u} + \mathbf{w}\times\mathbf{u}$$

$$(c\mathbf{v})\times\mathbf{w} = c(\mathbf{v}\times\mathbf{w})$$

Direction (up or down) is determined by the right-hand rule.

$$\boxed{\mathbf{v}\times\mathbf{w} = -\,\mathbf{w}\times\mathbf{v}}$$

unlike a product of numbers or dot product, vector product is not commutative!

Vector product

Coordinate expressions

$\mathbf{v} \times \mathbf{w}$ **is perpendicular to v, and w:**

$$(\mathbf{u} \cdot \mathbf{v}) = 0 \qquad (\mathbf{u} \cdot \mathbf{w}) = 0$$

the length of u is $|\mathbf{v}||\mathbf{w}|\,\mathbf{sin}\,\alpha$ **:**

$$(\mathbf{u} \cdot \mathbf{u}) = |\mathbf{v}|^2 |\mathbf{w}|^2\, \mathbf{sin}^2 \alpha = |\mathbf{v}|^2 |\mathbf{w}|^2 (1 - \mathbf{cos}^2 \alpha)$$

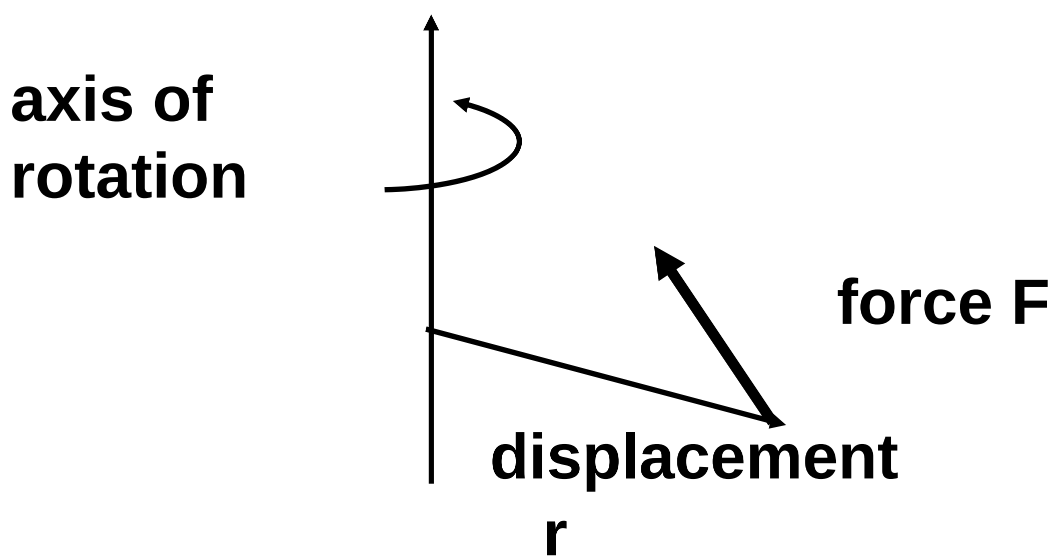
$$= |\mathbf{v}|\ |\mathbf{w}|\ \left(|\mathbf{v}||\mathbf{w}| - (\mathbf{v}, \mathbf{w})\right)$$

Solve three equations for u_x, u_y, u_z

Vector product

Physical interpretation: torque

torque = $\mathbf{r} \times \mathbf{F}$



Vector product

Coordinate expression:

$$\det\begin{bmatrix} e_x & e_y & e_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} = \left[\det\begin{bmatrix} v_y & v_z \\ w_y & w_z \end{bmatrix}, -\det\begin{bmatrix} v_x & v_z \\ w_x & w_z \end{bmatrix}, \det\begin{bmatrix} v_x & v_y \\ w_x & w_y \end{bmatrix}\right]$$

Notice that if $v_z = w_z = 0$, that is, vectors are 2D, the cross product has only one nonzero component (z) and its length is the determinant

$$\det\begin{bmatrix} v_x & v_y \\ w_x & w_y \end{bmatrix}$$

Vector product

More properties

$$(\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$$

$$((\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})$$