Additional notes for assignment 1: kinematics of a turning car

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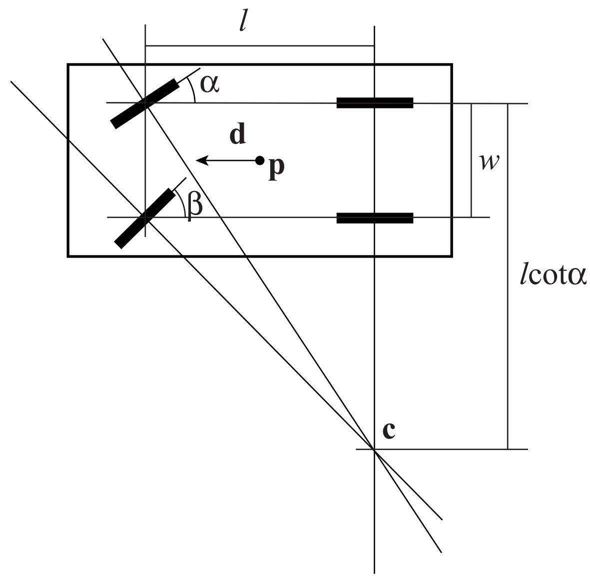
The position of the car is given by the position of its center $\mathbf{p}$ and the unit vector $\mathbf{d}$ along the body of the car.

Suppose at some moment of time t the car is moving with speed s, and the outer (the one that is further away from the center of rotation) front wheel is rotated by α (see the figure below). We need to determine how to calculate the position of the center of the car and the vector $\mathbf{d}$ at a moment $t + dt$, for a small time step dt.



To do this we rely on the following general fact: at any moment in time, the motion of the car can be regarded as a rotation around a center (this center can be at infinity if the car is not turning).

To find this center, we observe that if there is no sliding, the wheels move in the direction perpendicular to their axes. As the velocity of any point of a rotating object is perpendicular to the line connecting the point with the center of rotation, this means that the center of rotation is the intersection of the axes of all four wheels.

One can observe that while the rear wheels have the same axis, the front wheels, if rotated, have separate axes. If the wheels are rotated by the same angle, the lines along their axes do not intersect the axis of the rear wheels at the same point. This means that the front wheels have to be rotated by different angles, α and β. This is what is actually done in real cars.

We calculate the increments in three steps: first, we find the current center of rotation $\mathbf{c}$; next we compute the current angular velocity ω, and finally we rotate the point $\mathbf{p}$ around $\mathbf{c}$ by ωdt.

1. The distance from the center of the outer rear wheel to the center of rotation is $l \cot \alpha$. One can express its world-coordinate position using vector $\mathbf{d}$ and the perpendicular vector $\mathbf{d}^{\perp}$. From the figure one can see that the position of the center of the outer rear wheel is $\mathbf{p}^r = \mathbf{p} - (l/2)\mathbf{d} - (w/2)\mathbf{d}^{\perp}$. The center of rotation is displaced from it in the direction of $\mathbf{d}^{\perp}$ by $l \cot \alpha$, so

$$\mathbf{c} = \mathbf{p}^r + l \cot \alpha \mathbf{d}^{\perp}$$

2. Assuming that the speed of the outer rear wheel is s, and using the fact that the speed of a point at distance r from the center of rotation is ωr, we get $\omega = s/(l \cot \alpha)$.

3. Finally, we apply the 2D rotation matrix $R(\omega dt)$ for the angle ωdt to vectors $\mathbf{d}$ and $\mathbf{r} = \mathbf{p} - \mathbf{c}$. The new position of $\mathbf{p}$ is $\mathbf{c} + R(\omega dt)\mathbf{r}$.

Given $\mathbf{p}$ and $\mathbf{d}$, you can draw the car in the correct orientation and position by drawing it at the origin, oriented along the x axis, and then rotating it by the angle $\texttt{atan2}(\mathbf{d}_\texttt{y}, \mathbf{d}_\texttt{x})$ and translating by $\mathbf{p}$.

In your code, the user may control the angle α directly; computing β is left as an exercise.