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DUAL-PRIMAL FETI METHODS FOR LINEAR ELASTICITY

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Abstract. Dual-Primal FETI methods are nonoverlapping domain decomposition methods where some of the continuity constraints across subdomain boundaries are required to hold throughout the iterations, as in primal iterative substructuring methods, while most of the constraints are enforced by Lagrange multipliers, as in one-level FETI methods. The purpose of this article is to develop strategies for selecting these constraints, which are enforced throughout the iterations, such that good convergence bounds are obtained, which are independent of even large changes in the stiffnesses of the subdomains across the interface between them. A theoretical analysis is provided and condition number bounds are established which are uniform with respect to arbitrarily large jumps in the Young's modulus of the material and otherwise only depend polylogarithmically on the number of unknowns of a single subdomain.

Key words. domain decomposition, Lagrange multipliers, FETI, preconditioners, elliptic systems, elasticity, finite elements.

AMS subject classifications. 65F10,65N30,65N55

1. Introduction. We will consider iterative substructuring methods with Lagrange multipliers for the elliptic system of linear elasticity. The algorithms belong to the family of dual-primal FETI (finite element tearing and interconnection) methods which was introduced for linear elasticity problems in the plane in [8] and then extended to three dimensional elasticity problems in [9]. In dual-primal FETI (FETI-DP) methods, some continuity constraints on primal displacement variables are required to hold throughout the iterations, as in primal iterative substructuring methods, while most of the constraints are enforced by the use of dual Lagrange multipliers, as in the older one-level FETI algorithms. The primal constraints should be chosen so that the local problems become invertible. They also provide a coarse problem and they should be selected so that the iterative method converges rapidly. We also wish to use relatively few, and effective, primal constraints since the they represent a global part of the preconditioner which is relatively difficult to parallelize.

More recently, the family of algorithms for scalar elliptic problems in three dimensions was extended and a theory was provided in [15, 16]; see also [24, Section 6.4]. It was shown that the condition number of the dual-primal FETI methods can be bounded polylogarithmically as a function of the dimension of the individual subregion problems and that the bounds can otherwise be made independent of the number of subdomains, the mesh size, and jumps in the coefficients. In the case of the elliptic system of partial differential equations arising from linear elasticity, essential changes in the selection of the primal constraints have to be made in order to obtain the same quality bounds for elasticity problems as in the scalar case. Special emphasis will be given to developing robust condition number estimates with bounds which are independent of arbitrarily large jumps of the material coefficients. For benign coefficients, without large jumps, it is sufficient to select an appropriate set of edge averages as

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primal constraints to obtain good bounds, whereas for arbitrary coefficient distributions, additional primal first order moments and constraints at some of the vertices are also required. We note that there is extensive and ongoing experimental work on dual-primal FETI methods for linear elasticity; cf. [11]. The results obtained so far, using the algorithms developed in this paper, are quite promising.

We note that our results and strategies of selecting constraints immediately carry over to the more recently developed Neumann-Neumann methods with constraints, known as the BDDC algorithms; cf. [3, 18, 19]. This is so, since Mandel, Dohrmann, and Tezaur [19] have shown that for any given set of constraints, the BDDC and FETI-DP methods have almost all their eigenvalues in common; see also Fragakis and Papadrakakis [10] for earlier experimental work. We note that the results of [19] are obtained by algebra alone and that the authors do not address the question on how best to select the set of primal constraints.

The remainder of this article is organized as follows. In Section 2, we introduce the equations of linear elasticity in three dimensions and provide different Korn inequalities which are needed in our analysis. In Section 3, we introduce our domain decomposition and some geometric notation as well as the associated finite element spaces. In Section 4, we introduce our family of dual-primal FETI methods in an abstract setting and in Section 5, we establish certain conditions which will allow us to establish good bounds. In Section 6, we discuss two different possibilities of implementing our FETI-DP algorithms, one using global optional Lagrange multipliers and another where a change of basis is applied. In Section 7, we collect some auxiliary technical lemmas which are needed in our convergence analysis which is presented in Section 8. In a final subsection, we outline possible strategies of selecting a sufficiently rich set of constraints.

We note that some of the results of this paper have previously been presented in a conference article; cf. [14].

2. The equations of linear elasticity and Korn inequalities. The equations of linear elasticity model the displacement of a linear elastic material under the action of external and internal forces. The elastic body occupies a domain $\Omega \subset \mathbb{R}^3$, which is assumed to be polyhedral and of diameter one. We denote its boundary by $\partial\Omega$ and assume that one part of it, $\partial\Omega_D$, is clamped, i.e., with homogeneous Dirichlet boundary conditions, and that the rest, $\partial\Omega_N := \partial\Omega \setminus \partial\Omega_D$, is subject to a surface force $\mathbf{g}$, i.e., a natural boundary condition. We can also introduce a body force $\mathbf{f}$, e.g., gravity. With $\mathbf{H^1}(\Omega) := (H^1(\Omega))^3$, the appropriate space for a variational formulation is the Sobolev space $\mathbf{H_0^1}(\Omega, \partial\Omega_D) := \{\mathbf{v} \in \mathbf{H^1}(\Omega) : \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega_D\}$. The linear elasticity problem consists in finding the displacement $\mathbf{u} \in \mathbf{H_0^1}(\Omega, \partial\Omega_D)$ of the elastic body Ω, such that

$$(1)\int_\Omega G(\mathbf{x})\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) d\mathbf{x} + \int_\Omega G(\mathbf{x})\,\beta(\mathbf{x})\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x} = \langle \mathbf{F}, \mathbf{v}\rangle \;\; \forall \mathbf{v} \in \mathbf{H_0^1}(\Omega, \partial\Omega_D).$$

Here G and β are material parameters which depend on the Young's modulus $E > 0$ and the Poisson ratio $\nu \in (0, 1/2]$; we have $G = E/(1+\nu)$ and $\beta = \nu/(1-2\nu)$. (The coefficients are also referred to as the Lamé parameters.) In this article, we only consider the case of compressible elasticity, which means that the Poisson ratio ν is bounded away from 1/2. Furthermore, $\varepsilon_{ij}(\mathbf{u}) := \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the linearized strain tensor, and

$$\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) = \sum_{i,j=1}^{3} \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}), \quad \langle \mathbf{F}, \mathbf{v} \rangle := \int_\Omega \mathbf{f}^T \mathbf{v}\, d\mathbf{x} \;+\; \int_{\partial\Omega_N} \mathbf{g}^T \mathbf{v}\, d\sigma.$$

For convenience, we also introduce the notation

$$(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{L_2(\Omega)} := \int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) d\mathbf{x}.$$

The bilinear form associated with linear elasticity is then

$$a(\mathbf{u}, \mathbf{v}) = (G\, \varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{L_2(\Omega)} \;+\; (G\, \beta \operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v})_{L_2(\Omega)}.$$

We will also use the standard Sobolev space norm

$$\|\mathbf{u}\|_{H^1(\Omega)} := \left(|\mathbf{u}|^2_{H^1(\Omega)} + \|\mathbf{u}\|^2_{L_2(\Omega)} \right)^{1/2}$$

with $\|\mathbf{u}\|^2_{L_2(\Omega)} := \sum_{i=1}^{3} \int_\Omega |u_i|^2 d\mathbf{x}$, and $|\mathbf{u}|^2_{H^1(\Omega)} := \sum_{i=1}^{3} \|\nabla u_i\|^2_{L_2(\Omega)}$. We note that if we rescale our region by a dilation, the two terms of the full H^1-norm scale differently and we should introduce a factor $1/H^2$ in front of the square of the L_2-norm if the diameter of the region is on the order of H. It is obvious that the bilinear form $a(\cdot, \cdot)$ is continuous with respect to $\| \cdot \|_{H^1(\Omega)}$, although the bound depends on the Lamé parameters. Continuity follows from the elementary inequalities

$$\begin{aligned} |(\operatorname{div} (\mathbf{u}), \operatorname{div} (\mathbf{v}))_{L_2(\Omega_i)}| &\leq |\mathbf{u}|_{H^1(\Omega_i)} \, |\mathbf{v}|_{H^1(\Omega_i)} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H^1}(\Omega_i), \\ |(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{L_2(\Omega_i)}| &\leq |\mathbf{u}|_{H^1(\Omega_i)} \, |\mathbf{v}|_{H^1(\Omega_i)} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H^1}(\Omega_i). \end{aligned} \tag{2}$$

However, proving ellipticity is less trivial but it can be established from Korn's first inequality; see, e.g., Ciarlet [2].

Lemma 1 (Korn's first inequality). *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain. Then, there exists a positive constant $C = C(\Omega, \partial\Omega_D) > 0$, invariant under dilation, such that*

$$|\mathbf{u}|^2_{H^1(\Omega)} \leq C \, (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{u}))_{L_2(\Omega)} \quad \forall \mathbf{u} \in \mathbf{H^1_0}(\Omega, \partial\Omega_D).$$

The wellposedness of the linear system (1) follows immediately from the continuity and ellipticity of the bilinear form $a(\cdot, \cdot)$.

It follows from Korn's first inequality that $\|\varepsilon(\mathbf{u})\|_{L_2(\Omega)}$ is equivalent to $|\mathbf{u}|_{H^1(\Omega)}$ on $\mathbf{H^1_0}(\Omega, \partial\Omega_D)$. This result is not directly valid for the case of purely natural boundary conditions when we work with the entire space $\mathbf{H^1}(\Omega)$. This case is of interest when considering floating subregions, i.e., those that do not touch $\partial\Omega_D$. However, a Gårding inequality is provided by Korn's second inequality. This inequality will only be needed for our purposes on subdomains on which the Lamé parameters are assumed to be homogeneous, i.e., do not vary greatly.

Lemma 2 (Korn's second inequality). *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain of diameter one. Then, there exists a positive constant $C = C(\Omega)$, such that*

$$\|\mathbf{u}\|^2_{H^1(\Omega)} \leq C \left((\varepsilon(\mathbf{u}), \varepsilon(\mathbf{u}))_{L_2(\Omega)} + \|\mathbf{u}\|^2_{L_2(\Omega)} \right) \quad \forall \mathbf{u} \in \mathbf{H^1}(\Omega).$$

There are several proofs; see, e.g., Nitsche [22].

We can now derive a Korn inequality on the space $\{\mathbf{u} \in \mathbf{H^1}(\Omega) : \mathbf{u} \perp \mathbf{ker}\,(\varepsilon)\}$. The null space $\mathbf{ker}\,(\varepsilon)$ is the space of rigid body motions and orthogonality is defined with respect to the L_2-inner product. Thus, the linearized strain tensor of $\mathbf{u}$ and its divergence vanish only for the elements of the space spanned by the three translations

$$\mathbf{r}_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{r}_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{r}_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{3}$$

and the three rotations

$$\mathbf{r}_4 := \frac{1}{H}\begin{bmatrix} x_2 - \hat{x}_2 \\ -x_1 + \hat{x}_1 \\ 0 \end{bmatrix}, \mathbf{r}_5 := \frac{1}{H}\begin{bmatrix} -x_3 + \hat{x}_3 \\ 0 \\ x_1 - \hat{x}_1 \end{bmatrix}, \mathbf{r}_6 := \frac{1}{H}\begin{bmatrix} 0 \\ x_3 - \hat{x}_3 \\ -x_2 + \hat{x}_2 \end{bmatrix}. \tag{4}$$

Here $\hat{\mathbf{x}} \in \Omega$ and H denotes the diameter of Ω. The shift of the origin makes the basis for the space of rigid body modes well conditioned and the scaling and shift make the $L_2(\Omega)-$norms of these six functions scale in the same way with H. We will also use the notation $\mathbf{r}_k = (r_{kl})_{l=1,2,3}, k = 1, \ldots, 6$, with $r_{k\ell}$ the ℓ-th component of the k-th rigid body mode.

We now introduce two alternative inner products in $\mathbf{H^1}(\Omega)$, for a region Ω of diameter one,

$$(\mathbf{u}, \mathbf{v})_{E_1} := (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{L_2(\Omega)} + (\mathbf{u}, \mathbf{v})_{L_2(\Omega)}$$

and

$$(\mathbf{u}, \mathbf{v})_{E_2} := (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{L_2(\Omega)} + \sum_{i=1}^{6} (\mathbf{u}, \mathbf{r}_i)_{L_2(\Sigma)} (\mathbf{v}, \mathbf{r}_i)_{L_2(\Sigma)},$$

where

$$(\mathbf{u}, \mathbf{r}_i)_{L_2(\Sigma)} = \int_{\Sigma} \mathbf{u}^T \mathbf{r}_i dx. \tag{5}$$

Here, $\Sigma \subset \partial\Omega$ is assumed to have positive measure. By Lemma 2, $\| \cdot \|_{E_1}$, given by the inner product $(\cdot, \cdot)_{E_1}$, is a norm and so, by construction, is $\| \cdot \|_{E_2}$. These norms are equivalent:

Lemma 3. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain of diameter one and let $\Sigma \subset \partial\Omega$ be of positive measure. Then, there exist constants $0 < c \leq C < \infty$, such that*

$$c\,\|\mathbf{u}\|_{E_1} \leq \|\mathbf{u}\|_{E_2} \leq C\,\|\mathbf{u}\|_{E_1} \quad \forall \mathbf{u} \in \mathbf{H^1}(\Omega).$$

Proof. The proof of the right inequality follows immediately from the Cauchy–Schwarz inequality and a simple trace theorem. The left inequality is proven by contradiction and by using Rellich's theorem as in a proof of generalized Poincaré–Friedrichs inequalities, cf., e.g., Nečas [21, Chap. 2.7]. For such an argument, it is important that the linear functionals $l_i(\mathbf{u}) = (\mathbf{u}, \mathbf{r}_i)_{L_2(\Sigma)}$ be bounded on $\mathbf{H^1}(\Omega)$; this is a consequence of a Cauchy-Schwarz inequality and the same trace theorem.

□

Using analogous arguments, combined with Lemma 2, we obtain:

LEMMA 4. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain of diameter one and let $\Sigma \subset \partial\Omega$ be of positive measure. Then, there exists a positive constant $C > 0$, such that*

$$|\mathbf{u}|^2_{H^1(\Omega)} + \|\mathbf{u}\|^2_{L_2(\Sigma)} \leq C\, \left((\varepsilon(\mathbf{u}), \varepsilon(\mathbf{u}))_{L_2(\Omega)} + \|\mathbf{u}\|^2_{L_2(\Sigma)} \right) \quad \forall \mathbf{u} \in \mathbf{H^1}(\Omega).$$

Using (2) and Lemmas 2 and 3, in combination with a scaling argument, we obtain a Korn inequality on a subspace of $\mathbf{H^1}(\Omega)$.

LEMMA 5. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain. Then, there exists a constant $c > 0$, invariant under dilation, such that*

$$c\,|\mathbf{u}|_{H^1(\Omega)} \leq \|\varepsilon(\mathbf{u})\|_{L_2(\Omega)} \leq |\mathbf{u}|_{H^1(\Omega)},$$

where $\mathbf{u} \in \{\mathbf{v} \in \mathbf{H^1}(\Omega) : (\mathbf{v}, \mathbf{r})_{L_2(\Sigma)} = 0 \quad \forall \mathbf{r} \in \mathbf{ker}\,(\varepsilon)\}$.

In the following, we will make extensive use of trace spaces equipped with trace norms. We therefore recall some definitions. We will first consider scalar valued Sobolev spaces. Let Σ, again, be a subset of $\partial\Omega$ with positive measure. Then, a seminorm, which is equivalent to $|\cdot|_{H^{1/2}(\Sigma)}$ on $H^{1/2}(\Sigma)$, can be defined for $u \in H^{1/2}(\Sigma)$ as

$$\inf_{\substack{v \in H^1(\Omega) \\ v_{|\Sigma} = u}} |v|_{H^1(\Omega)}.$$

Clearly, $|\mathbf{u}|^2_{H^{1/2}(\Sigma)} := \sum_{i=1}^3 |u_i|^2_{H^{1/2}(\Sigma)}$ defines a seminorm on the product trace space $\mathbf{H^{1/2}}(\Sigma) := (H^{1/2}(\Sigma))^3$. Another useful seminorm on $\mathbf{H^{1/2}}(\Sigma)$ is given by

$$|\mathbf{u}|_{E(\Sigma)} := \inf_{\substack{\mathbf{v} \in \mathbf{H^1}(\Omega) \\ \mathbf{v}_{|\Sigma} = \mathbf{u}}} \|\varepsilon(\mathbf{v})\|_{L_2(\Omega)}.$$

We will denote by $\mathbf{u}_{harm}$ and $\mathbf{u}_{elast} \in \{\mathbf{v} \in \mathbf{H^1}(\Omega) : \mathbf{v}_{|\Sigma} = \mathbf{u}\}$ the harmonic and elastic extension of $\mathbf{u}$, respectively, defined by $|\mathbf{u}_{harm}|_{H^1(\Omega)} = |\mathbf{u}|_{H^{1/2}(\Sigma)}$ and $\|\varepsilon(\mathbf{u}_{elast})\|_{L_2(\Omega)} = |\mathbf{u}|_{E(\Sigma)}$. From Lemma 4, we immediately see that for $\mathbf{u} \in \mathbf{H^{1/2}}(\Sigma)$

$$\begin{aligned} |\mathbf{u}|^2_{H^{1/2}(\Sigma)} + \|\mathbf{u}\|^2_{L_2(\Sigma)} &\leq |\mathbf{u}_{elast}|^2_{H^1(\Omega)} + \|\mathbf{u}\|^2_{L_2(\Sigma)} \\ &\leq 1/c\, \left(\|\varepsilon(\mathbf{u}_{elast})\|^2_{L_2(\Omega)} + \|\mathbf{u}\|^2_{L_2(\Sigma)} \right) \\ &= 1/c\, \left(|\mathbf{u}|^2_{E(\Sigma)} + \|\mathbf{u}\|^2_{L_2(\Sigma)} \right). \end{aligned} \tag{6}$$

Using these estimates and a standard scaling argument, we also have a Korn inequality on the trace space $\mathbf{H^{1/2}}(\Sigma)$.

LEMMA 6. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain of diameter H and $\Sigma \subset \partial\Omega$ be an open subset with positive surface measure. Then, there exists a constant $C > 0$, invariant under dilation, such that*

$$|\mathbf{u}|^2_{H^{1/2}(\Sigma)} + \frac{1}{H}\|\mathbf{u}\|^2_{L_2(\Sigma)} \leq C\, \left(|\mathbf{u}|^2_{E(\Sigma)} + \frac{1}{H}\|\mathbf{u}\|^2_{L_2(\Sigma)} \right)$$

where $\mathbf{u} \in \mathbf{H^{1/2}}(\Sigma)$.

We also have the following Korn inequality.

LEMMA 7. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain of diameter H and $\Sigma \subset \partial\Omega$ be an open subset with positive surface measure. There exists a positive constant C, independent of H, such that*

$$\inf_{\mathbf{r}\in\mathbf{ker}\,(\varepsilon)} \|\mathbf{u}-\mathbf{r}\|^2_{L_2(\Sigma)} \leq C\,H\,|\mathbf{u}|^2_{E(\Sigma)} \quad \forall \mathbf{u} \in \mathbf{H^{1/2}}(\Sigma).$$

Proof. We first prove the lemma for a domain Ω of unit diameter. Let $\mathbf{u} \in \mathbf{H^{1/2}}(\Sigma)$ be arbitrary but fixed and let $\mathbf{r} \in \mathbf{ker}\,(\varepsilon)$ be the minimizing rigid body mode, for which $(\mathbf{u}-\mathbf{r},\mathbf{r}_i)_{L_2(\Sigma)} = 0, i = 1,\ldots,6$. Then, by using a standard trace theorem, and Lemmas 2 and 3, we obtain

$$\begin{aligned}
\|\mathbf{u}-\mathbf{r}\|^2_{L_2(\Sigma)} &\leq C\,(|(\mathbf{u}-\mathbf{r})_{elast}|^2_{H^1(\Omega)} + \|(\mathbf{u}-\mathbf{r})_{elast}\|^2_{L_2(\Omega)}) \\
&\leq C\,(|\mathbf{u}-\mathbf{r}|^2_{E(\Sigma)} + \|(\mathbf{u}-\mathbf{r})_{elast}\|^2_{L_2(\Omega)}) \\
&\leq C\,(|\mathbf{u}-\mathbf{r}|^2_{E(\Sigma)} + \sum_{i=1}^{6}((\mathbf{u}-\mathbf{r},\mathbf{r}_i)_{L_2(\Sigma)})^2) \\
&= C\,|\mathbf{u}|^2_{E(\Sigma)}.
\end{aligned}$$

We now obtain the result using a standard scaling argument.

□

3. Finite elements and geometry. We will only consider compressible elastic materials. It then follows from Lemma 1 that the bilinear form $a(\cdot,\cdot)$ is uniformly elliptic and uniformly continuous. It is therefore sufficient to discretize our elliptic problem (1) by low order, conforming finite elements, e.g., linear or trilinear elements.

Let us assume that a triangulation τ^h of Ω is given which is shape regular and has a typical diameter of h. We denote by $\mathbf{W}^h := \mathbf{W}^h(\Omega) \subset \mathbf{H}^1_0(\Omega,\partial\Omega_D)$ the corresponding conforming finite element space of finite element functions. The corresponding discrete problem is then

$$a(\mathbf{u}_h,\mathbf{v}_h) = \langle \mathbf{F},\mathbf{v}_h\rangle \quad \forall \mathbf{v}_h \in \mathbf{W}^h. \tag{7}$$

When there is no risk of confusion, we will drop the subscript h.

Let the domain $\Omega \subset \mathbb{R}^3$ be decomposed into nonoverlapping subdomains $\Omega_i, i = 1,\ldots,N$, each of which is the union of finite elements with matching finite element nodes on the boundaries of neighboring subdomains across the interface Γ. The interface Γ is the union of subdomain faces, edges, all of them regarded as open sets, and subdomain vertices. Faces are sets which are shared by two subregions, edges normally by more than two subregions, and vertices are endpoints of edges. Vectors of interior variables will be equipped with the subscript I. We denote the faces of Ω_i by $\mathcal{F}^{ij}$, its edges by $\mathcal{E}^{ik}$, and its vertices by $\mathcal{V}^{il}$. Subdomain vertices that lie on $\partial\Omega_N$ are part of Γ, while subdomain faces that are part of $\partial\Omega_N$ are not; the nodes on those faces will always be treated as interior. If Γ intersects $\partial\Omega_N$ along an edge common to the boundaries of only two subdomains, we will normally regard it as part of the face common to this pair of subdomains; if there are more than two subdomains, it will be regarded as an edge of Γ. Similarly, we will regard a subdomain vertex on $\partial\Omega_N$ part of an interior edge unless there are several such edges that end at the vertex. In the latter case, we treat the vertex the same way as a vertex in the interior of the domain. We note that any subdomain the boundary of which does not intersect $\partial\Omega_D$

is a floating subdomain, i.e., a subdomain for which only natural boundary conditions are imposed.

These geometrical entities can also be defined in terms of certain equivalence classes. Let us denote the sets of nodes on $\partial\Omega, \partial\Omega_i$, and Γ by $\partial\Omega_h, \partial\Omega_{i,h}$, and Γ_h, respectively. For any interface nodal point $x \in \Gamma_h$, we define

$$\mathcal{N}_x := \{j \in \{1, \ldots, N\} : x \in \partial\Omega_j\},$$

i.e., $\mathcal{N}_x$ is the set of indices of all subdomains with x in the closure of the subdomain.

Associated with the nodes of the finite element mesh, we have a graph, the nodal graph, which represents the node-to-node adjacency. For a given node $x \in \Gamma_h$, we denote by $\mathcal{C}_{con}(x)$ the connected component of the nodal subgraph, defined by $\mathcal{N}_x$, to which x belongs. For two interface points $x, y \in \Gamma_h$, we introduce an equivalence relation by

$$x \sim y :\Longleftrightarrow \mathcal{N}_x = \mathcal{N}_y \text{ and } y \in \mathcal{C}_{con}(x).$$

We can now describe faces, edges, and vertices using their equivalence classes. Here, $|G|$ denotes the cardinality of the set G. We find that

$$\begin{array}{rcl} x \in \mathcal{F} & :\Longleftrightarrow & |\mathcal{N}_x| = 2 \\ x \in \mathcal{E} & :\Longleftrightarrow & |\mathcal{N}_x| \geq 3 \text{ and } \exists y \in \Gamma_h, y \neq x, \text{ such that } y \sim x \\ x \in \mathcal{V} & :\Longleftrightarrow & |\mathcal{N}_x| \geq 3 \text{ and } \nexists y \in \Gamma_h, \text{ such that } x \sim y. \end{array}$$

In our theoretical analysis, we assume that each subregion Ω_i is the union of a number of shape regular tetrahedral coarse elements and that the number of such tetrahedra is uniformly bounded for each subdomain. Therefore, the subregions are not very thin and we can also easily show that the diameters of any pair of neighboring subdomains are comparable. In such a case, our definition of faces, edges, and vertices conform with our basic geometric intuition. On the other hand, for subdomains generated by an automatic mesh partitioner, the situation can be quite complicated. We can, e.g., have several edges with the same index set $\mathcal{N}_x$ or an edge and a vertex with the same $\mathcal{N}_x$. In practice, we can also have situations when there are not enough edges and potential edge constraints for some subdomains. Then, we have to use constraints on some extra edges on $\partial\Omega_N$, which otherwise would be regarded as part of a face; see above.

We denote the standard finite element space of continuous, piecewise linear functions on Ω_i by $\mathbf{W}^h(\Omega_i)$; we always assume that these functions vanish on $\partial\Omega_D$. For simplicity, we assume that the triangulation of each subdomain is quasi uniform. The diameter of Ω_i is H_i, or generically, H. We denote the corresponding finite element trace spaces by $\mathbf{W}^{(i)} := \mathbf{W}^h(\partial\Omega_i \cap \Gamma), i = 1, \ldots, N$, and by $\mathbf{W} := \prod_{i=1}^N \mathbf{W}^{(i)}$ the associated product space. We will often consider elements of $\mathbf{W}$, which are discontinuous across the interface.

For each subdomain Ω_i, we define the local stiffness matrix $K^{(i)}$, which we view as an operator on $\mathbf{W}^h(\Omega_i)$. On the product space $\prod_{i=1}^N \mathbf{W}^h(\Omega_i)$, we define the operator K as the direct sum of the local stiffness operators $K^{(i)}$, i.e.,

$$K := \bigoplus_{i=1}^{N} K^{(i)}. \tag{8}$$

In an implementation, K corresponds to a block diagonal matrix since, so far, there is no coupling across the interface. The finite element approximation of the elliptic problem is continuous across Γ and we denote the corresponding subspace of $\mathbf{W}$ by $\widehat{\mathbf{W}}$. We note that while the stiffness matrix K and its Schur complement, obtained after eliminating the variables interior to the subregions, which corresponds to the product space $\mathbf{W}$, are singular if there are any floating subdomains, those of $\widehat{\mathbf{W}}$ are not.

In the present study, as in others of FETI–DP methods, we also work with subspaces $\widetilde{\mathbf{W}} \subset \mathbf{W}$ for which sufficiently many constraints are enforced so that the resulting leading diagonal block matrix of the FETI saddle point problem, to be introduced in (20), though no longer block diagonal, is strictly positive definite. These are called primal constraints and in our discussion they usually consist of certain edge averages and first order moments, which have common values across the interface of neighboring subdomains, and possibly of constraints at well chosen subdomain vertices (or other nodes), for which a partial subassembly is carried out. One of the benefits of working in $\widetilde{\mathbf{W}}$, rather than in $\mathbf{W}$, will be that certain related Schur complements, $\widetilde{S}_\varepsilon$ and S_ε, are strictly positive definite; cf. (10) and (12).

We further introduce two subspaces, $\widehat{\mathbf{W}}_\Pi \subset \widehat{\mathbf{W}}$ and $\widetilde{\mathbf{W}}_\Delta$, corresponding to a primal and a dual part of the space $\widetilde{\mathbf{W}}$. These subspaces play an important role in the description and analysis of our iterative method. We note that the dual subspace $\widetilde{W}_\Delta$ will be directly associated with jumps across the interface and with the Lagrange multipliers that are introduced to eliminate these jumps. The direct sum of these spaces equals $\widetilde{\mathbf{W}}$, i.e.,

$$\widetilde{\mathbf{W}} = \widehat{\mathbf{W}}_\Pi \oplus \widetilde{\mathbf{W}}_\Delta. \tag{9}$$

The second subspace, $\widetilde{\mathbf{W}}_\Delta$, is the direct sum of local subspaces $\widetilde{\mathbf{W}}_\Delta^{(i)}$ of $\widetilde{\mathbf{W}}$, where each subdomain Ω_i contributes a subspace $\widetilde{\mathbf{W}}_\Delta^{(i)}$; only its i-th component in the sense of the product space $\widetilde{\mathbf{W}}$ is non trivial.

We now define certain Schur complement operators by using a variational formulation; for a matrix representation, see Section 6. Here, $\langle \cdot, \cdot \rangle$ will denote the ℓ_2-inner product. We first define Schur complement operators $S_\varepsilon^{(i)}, i = 1, \ldots, N$, operating on $\mathbf{W}^{(i)}$, by

$$\langle S_\varepsilon^{(i)} \mathbf{w}^{(i)}, \mathbf{w}^{(i)} \rangle = \min \langle K^{(i)} \mathbf{v}^{(i)}, \mathbf{v}^{(i)} \rangle \ \forall \mathbf{w}^{(i)} \in \mathbf{W}^{(i)}, \tag{10}$$

where we take the minimum over all $\mathbf{v}^{(i)} \in \mathbf{W}^h(\Omega_i)$ with $\mathbf{v}^{(i)}{}_{|\Gamma} = \mathbf{w}^{(i)}$. We can now define the Schur complement S_ε operating on $\mathbf{W}$ as the direct sum of the local Schur complements

$$S_\varepsilon := \bigoplus_{i=1}^{N} S_\varepsilon^{(i)}. \tag{11}$$

Next, we introduce a positive definite Schur complement $\widetilde{S}_\varepsilon$, operating on $\widetilde{\mathbf{W}}_\Delta$, by a variational problem: for all $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$,

$$\langle \widetilde{S}_\varepsilon \mathbf{w}_\Delta, \mathbf{w}_\Delta \rangle = \min_{\mathbf{w}_\Pi \in \widehat{\mathbf{W}}_\Pi} \langle S_\varepsilon (\mathbf{w}_\Delta + \mathbf{w}_\Pi), \mathbf{w}_\Delta + \mathbf{w}_\Pi \rangle. \tag{12}$$

We note that any Schur complement of a positive definite, symmetric matrix is always associated with such a variational problem. We also obtain, analogously, a reduced right hand side $\tilde{\mathbf{f}}_\Delta$, from the load vectors associated with the individual subdomains.

We now consider the relation between the Schur complement of the elasticity stiffness matrix, S_ε, and S the one arising from discretizing a vector valued Laplace operator scaled by the values of G. Obviously, S can also be defined as the direct sum of local Schur complements

$$S := \bigoplus_{i=1}^{N} S^{(i)}, \tag{13}$$

where the $S^{(i)}$ are again given by a variational argument as in (10), using the discrete, scaled, vector valued Laplace operator instead of $K^{(i)}$. We furthermore introduce bilinear forms which represent the contributions of the individual subdomains to the bilinear from $a(\mathbf{u}, \mathbf{v})$:

$$a(\mathbf{u}, \mathbf{v}) := \sum_{i=1}^{N} a_i(\mathbf{u}, \mathbf{v})$$

with $G_i := \frac{E_i}{1+\nu_i}, \beta_i := \frac{\nu_i}{1-2\nu_i}$, and

$$a_i(\mathbf{u}, \mathbf{v}) := G_i \left((\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{L_2(\Omega_i)} + \beta_i (\mathrm{div}\,(\mathbf{u}), \mathrm{div}\,(\mathbf{v})_{L_2(\Omega_i)} \right).$$

We will assume that G_i and β_i are constant on the subdomain Ω_i. We obviously have for $\mathbf{u} \in \mathbf{W}^h$.

$$|\mathbf{u}|^2_{S_\varepsilon} \leq \max_{i=1,\ldots,N} (1+\beta_i)\, |\mathbf{u}|^2_S. \tag{14}$$

In order to define certain scaling operators, we need to define weighted counting functions δ_i for each subdomain Ω_i. These are functions in the scalar finite element trace space on $\partial\Omega_i$. They are defined, for $\gamma \in [1/2, \infty)$, by

$$\delta_i(x) := \frac{\sum_{j \in \mathcal{N}_x} G_j^\gamma(x)}{G_i^\gamma(x)}, \quad x \in \partial\Omega_{i,h} \cap \Gamma_h. \tag{15}$$

Here, as before, $\mathcal{N}_x$ is the set of indices of the subregions which have x on its boundary. This formula can also be used for material coefficients G_i which vary over the boundary of the subdomains but in our theory, we only consider the case when the coefficients are constant in each subdomain. We note that any node of Γ_h belongs either to a face common to two subdomains, to an edge common to at least three subregions, or is a vertex of several substructures. The pseudo inverses $\delta_i^\dagger$ are defined as

$$\delta_i^\dagger(x) = \delta_i^{-1}(x), \quad x \in \partial\Omega_{i,h} \cap \Gamma_h. \tag{16}$$

We further introduce an extension operators $R_i^T : \mathbf{W}^{(i)} \to \widehat{\mathbf{W}}$, such that the continuous global function $R_i^T \mathbf{w}_i \in \widehat{\mathbf{W}}$ shares the nodal values with $\mathbf{w}_i$ on $\partial\Omega_{i,h} \cap \Gamma_h$ and vanishes at all other nodes of Γ_h. We note that these functions provide a partition of unity:

$$\sum_i R_i^T (\delta_i^\dagger(x) \mathbf{1}) \equiv \mathbf{1} \quad \forall x \in \Gamma_h, \tag{17}$$

where $\mathbf{1} \in \widehat{\mathbf{W}}$ is the vector valued function with components equal to one at every point of Γ_h.

For $\gamma \geq 1/2$, we can easily show that

$$G_i (\delta_k^\dagger)^2 \leq \min(G_i, G_k). \tag{18}$$

4. The dual-primal FETI method. We reformulate the original finite element problem, reduced to the degrees of freedom of the second subspace $\widetilde{\mathbf{W}}_\Delta$, as a minimization problem with constraints given by the requirement of continuity across all of Γ_h: find $\mathbf{u}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, such that

$$\left.\begin{array}{c} J(\mathbf{u}_\Delta) := \frac{1}{2}\langle \widetilde{S}_\varepsilon \mathbf{u}_\Delta, \mathbf{u}_\Delta\rangle - \langle \tilde{\mathbf{f}}_\Delta, \mathbf{u}_\Delta\rangle \to \min \\ B_\Delta \mathbf{u}_\Delta = 0 \end{array}\right\}. \tag{19}$$

The jump operator B_Δ operates on $\widetilde{\mathbf{W}}$ and enforces pointwise continuity of the dual displacement degrees of freedom. At possible primal vertices, continuity is already enforced by subassembly and a jump operator applied to a function from $\widetilde{\mathbf{W}}$ would automatically be zero at these special degrees of freedom.

By introducing a set of Lagrange multipliers $\boldsymbol{\lambda} \in \mathbf{V} := \mathbf{range}\,(B_\Delta)$, to enforce the constraints $B_\Delta \mathbf{u}_\Delta = 0$, we obtain a saddle point formulation of (19):

$$\begin{bmatrix} \widetilde{S}_\varepsilon & B_\Delta^T \\ B_\Delta & O \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Delta \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{f}}_\Delta \\ \mathbf{0} \end{bmatrix}. \tag{20}$$

We note that we can add any element from $\mathbf{ker}\,(B_\Delta^T)$ to $\boldsymbol{\lambda}$ without changing the displacement solution $\mathbf{u}_\Delta$.

Since $\widetilde{S}_\varepsilon$ is invertible, we can eliminate $\mathbf{u}_\Delta$ and obtain the following system for the Lagrange multiplier variables:

$$F\boldsymbol{\lambda} = \mathbf{d}. \tag{21}$$

Here, our new system matrix F is defined by

$$F := B_\Delta \widetilde{S}_\varepsilon^{-1} B_\Delta^T \tag{22}$$

and the new right hand side by $\mathbf{d} := B_\Delta \widetilde{S}_\varepsilon^{-1} \tilde{\mathbf{f}}_\Delta$. Algorithmically, $\widetilde{S}_\varepsilon$ is only needed in terms of $\widetilde{S}_\varepsilon^{-1}$ times a vector $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$ and such an operation can be excecuted relatively inexpensively; see Section 6. The operator F will obviously depend on the choice of the subspaces $\widehat{\mathbf{W}}_\Pi$ and $\widetilde{\mathbf{W}}_\Delta$.

To define the FETI-DP Dirichlet preconditioner, we need to introduce scaled jump operators

$$B_{D,\Delta} := [B_{D,\Delta}^{(1)}, \ldots, B_{D,\Delta}^{(N)}].$$

Here, the $B_{D,\Delta}^{(i)}$ are defined as follows: each row of $B_\Delta^{(i)}$ with a nonzero entry corresponds to a Lagrange multiplier connecting the subdomain Ω_i with a neighboring subdomain Ω_j at a point $x \in \partial\Omega_{i,h} \cap \partial\Omega_{j,h}$. Multiplying each such row of $B_\Delta^{(i)}$ with $\delta_j^\dagger(x)$ gives us $B_{D,\Delta}^{(i)}$.

As in Klawonn and Widlund [13, Section 5], we solve the dual system (21) using the preconditioned conjugate gradient algorithm with the preconditioner

$$M^{-1} := P B_{D,\Delta} S_\varepsilon B_{D,\Delta}^T P^T, \tag{23}$$

where P is the ℓ_2-orthogonal projection from $\mathbf{range}\,(B_{D,\Delta})$ onto $\mathbf{V} = \mathbf{range}\,(B_\Delta)$, i.e., P removes the component from $\mathbf{ker}\,(B_\Delta^T)$ of an element in $\mathbf{range}\,(B_{D,\Delta})$. We note that P and P^T are only needed for the theoretical analysis to guarantee that the preconditioned residuals will belong to $\mathbf{V}$; cf. remark after (20). The projections can be dropped in the implementation; cf. the argument at the end of this section.

This definition of M^{-1} clearly depends on the choice of the subspaces $\widehat{\mathbf{W}}_\Pi$ and $\widetilde{\mathbf{W}}_\Delta$.

The FETI-DP method is the standard preconditioned conjugate gradient algorithm for solving the preconditioned system

$$M^{-1}F\boldsymbol{\lambda} = M^{-1}\mathbf{d}.$$

ALGORITHM 1.
(i) Initialization: $\mathbf{r}^0 := \mathbf{d} - F\boldsymbol{\lambda}^0$
(ii) Iterate for $k = 1, 2, \ldots,$ *until convergence,*

$$\begin{aligned}
\mathbf{z}^{k-1} &:= M^{-1}\mathbf{r}^{k-1} \\
\beta^k &:= \frac{\langle \mathbf{z}^{k-1}, \mathbf{r}^{k-1}\rangle}{\langle \mathbf{z}^{k-2}, \mathbf{r}^{k-2}\rangle} \quad [\beta^1 := 0] \\
\mathbf{p}^k &:= \mathbf{z}^{k-1} + \beta^k \mathbf{p}^{k-1} \quad [\mathbf{p}^1 := \mathbf{z}^0] \\
\alpha^k &:= \frac{\langle \mathbf{z}^{k-1}, \mathbf{r}^{k-1}\rangle}{\langle \mathbf{p}^k, F\mathbf{p}^k\rangle} \\
\boldsymbol{\lambda}^k &:= \boldsymbol{\lambda}^{k-1} + \alpha^k \mathbf{p}^k \\
\mathbf{r}^k &:= \mathbf{r}^{k-1} - \alpha^k F\mathbf{p}^k
\end{aligned}$$

We note that in the implementation of our preconditioner M^{-1}, we can drop the projection operator P and its transpose as can be seen by the following argument. Applying $B_{D,\Delta}S_\varepsilon B_{D,\Delta}^T$ to an element from $\mathbf{V}$ results in a vector μ which can be written as a sum $\mu = \mu_0 + \mu_1$ of components $\mu_0 \in \mathbf{ker}\,(B_\Delta^T)$ and $\mu_1 \in \mathbf{V} = \mathbf{range}\,(B_\Delta)$. When F is applied to μ, the component $F\mu_0$ vanishes and we also have $F\mu \in \mathbf{V}$. Examining Algorithm 1, we can also easily see that dropping P and P^T only affects the computed Lagrange multiplier solution but not the computed displacements. The residuals $\mathbf{r}^k$ are always in $\mathbf{V}$ and it is easy to show that the α_k and β_k are not affected.

5. Selection of constraints. In order to control the rigid body modes of a subregion, we need at least six constraints. To get an understanding of the type of primal constraints that are required to make our preconditioner effective, it is useful to examine two special cases.

In the first, we assume that we have two subdomains made of the same material, which have a face in common and are surrounded by subdomains made of a material with much smaller Young's modulus E. Such a problem will clearly have six low energy modes corresponding to the rigid body modes of the union of the two special subdomains. Any preconditioner that has less than six linearly independent primal constraints across that face will have at least seven low energy modes and will be far from spectrally equivalent to the original finite element model.

In the second case, we again consider two subdomains surrounded by subdomains with much smaller stiffnesses, i.e., Young's moduli. We now assume that the two special subdomains share only an edge. In this case, there are seven low energy modes of the finite element model corresponding to the same rigid body modes as before and an additional one. The new mode corresponds to a relative rotation of the two subdomains around their common edge. We conclude that in such a case, we should introduce five linearly independent primal constraints related to the special edge.

In the convergence theory presented in Section 8, we will first assume that the requirement of the first special case is met for each face, i.e., there are at least six linearly independent edge constraints for each face of the interface. Any such face will be called *fully primal*, cf. Definition 1. We note that any such edge constraint will serve as a constraint for every face adjacent to the edge in question. We do not have to make every face fully primal but, for every face, we have to have an *acceptable face path*, cf. Definition 3 and Section 8.3. We also note that using only constraints based on averages over faces might not always lead to a robust algorithm with respect to jumps in the stiffnesses of different materials; see [16, 12].

For coefficient distributions with only modest jumps across the interface Γ and for some special decompositions, we are able to work exclusively with edge averages; cf. Section 8.1. To be able to treat general coefficient distributions with arbitrarily large jumps, we also need first order moments, in addition to the averages, on certain edges such as those in the second special case discussed above. All these constraints will be written in terms of inner products of rigid body modes and the displacement over individual edges. There will be only five linearly independent constraints of this type since, restricted to an edge, one rotational rigid body mode is always linearly dependent of the others. This can be seen easily by a direct computation or by a change of coordinates such that the chosen edge coincides with the x_1-axis of the Cartesian coordinate system; then the third rotation $\mathbf{r}_6$ will vanish and the relevant first order moments are with respect to the second and third displacement components. Such an edge with three edge average constraints and two first order moment constraints will be called *fully primal*, cf. Definition 2. An edge will be called *primal* if there is at least one constraint, expressed in terms of an average, for at least one of its displacement components. As with the fully primal faces, we do not have to make every edge fully primal. Instead, we can make sure that there is an acceptable face path; cf. Definition 3. Finally, to be able to treat the most general distribution of coefficients, it can be necessary to make some vertices primal and we need the concept of an *acceptable vertex path*; cf. Definition 4.

Definition 1 (Fully primal face). *Let $\mathcal{F}^{ij}$ be a face. A set $f_m, m = 1, \ldots, 6$, of linearly independent linear functionals on $\mathbf{W}^{(i)}$ is called a set of primal constraint functionals on that face if it has the following properties:*

(i) $|f_m(\mathbf{w}^{(i)})|^2 \leq C\, H_i^{-1}(1 + \log(H_i/h_i))\{|\mathbf{w}^{(i)}|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_i}\|\mathbf{w}^{(i)}\|^2_{L_2(\mathcal{F}^{ij})}\}$

(ii) $f_m(\mathbf{r}_l) = \delta_{ml} \quad \forall m, l = 1, \ldots, 6, \quad \mathbf{r}_l \in \mathbf{ker}\,(\varepsilon)$.

Such a face is called a fully primal face.

We will sometimes write $f_m^{\mathcal{F}^{ij}}$ instead of f_m. As an example of functionals f_m, as considered in Definition 1, we can use appropriately chosen linear combinations of certain edge averages, g_n, of components of the displacement,

$$g_n(\mathbf{w}^{(i)}) := \frac{\int_{\mathcal{E}^{ik}} w_\ell^{(i)} dx}{\int_{\mathcal{E}^{ik}} 1 dx}, \;\; n = 1, \ldots, 6,$$

for a function $\mathbf{w}^{(i)} = (w_1^{(i)}, w_2^{(i)}, w_3^{(i)}) \in \mathbf{W}^{(i)}$ and appropriately chosen edges $\mathcal{E}^{ik}$ which belong to the boundary of the face $\mathcal{F}^{ij}$. We can show that in order to obtain six linearly independent linear functionals associated with a rectangular face $\mathcal{F}^{ij}$, we have to work with at least three different edges $\mathcal{E}^{ik}$.

The functionals $g_1, \ldots, g_6$, provide a basis of the dual space $(\mathbf{ker}\,(\varepsilon))'$. There also exists a dual basis of $(\mathbf{ker}\,(\varepsilon))'$, which we denote by $f_1, \ldots, f_6$, defined by $f_m(\mathbf{r}_l) = \delta_{ml}, m, l = 1, \ldots, 6$; thus, there exist $\beta_{lk} \in \mathbb{R}, l, k = 1, \ldots, 6$, such that for $\mathbf{w} \in \mathbf{W}^{(i)}$, we have

$$f_m(\mathbf{w}) = \sum_{n=1}^{6} \beta_{mn} g_n(\mathbf{w}), \quad m = 1, \ldots, 6. \tag{24}$$

Using a Cauchy-Schwarz inequality, we obtain

$$|g_m(\mathbf{w}^{(i)})|^2 \le C\, H_i^{-1}\, \|\mathbf{w}^{(i)}\|^2_{L_2(\mathcal{E}^{ik})}.$$

We can then show, by using Lemma 11, that

$$\|\mathbf{w}^{(i)}\|^2_{L_2(\mathcal{E}^{ik})} \le C\,(1 + \log(H_i/h_i))\,(|\mathbf{w}^{(i)}|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_i}\|\mathbf{w}^{(i)}\|^2_{L_2(\mathcal{F}^{ij})}).$$

Thus, the first requirement of Definition 1 is satisfied for the functionals f_m.

It is also possible to construct five of the functionals f_m in Definition 1 using five constraints, three averages and two first order moments, on one single edge. Before we discuss this possibility, we introduce the definition of a fully primal edge.

Definition 2 (Fully primal edge). *Let $\mathcal{F}^{ij}$ be a face and $\mathcal{E}^{ik}$ an edge which belongs to the boundary of $\mathcal{F}^{ij}$. A set $f_m, m = 1, \ldots, 5,$ of linearly independent linear functionals on $\mathbf{W}^{(i)}$ is called a set of primal constraint functionals on the edge $\mathcal{E}^{ik}$ if it has the following properties:*

(i) $|f_m(\mathbf{w}^{(i)})|^2 \le C\, H_i^{-1}(1 + \log(H_i/h_i))\{|\mathbf{w}^{(i)}|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_i}\|\mathbf{w}^{(i)}\|^2_{L_2(\mathcal{F}^{ij})}\}$

(ii) $f_m(\mathbf{r}_l) = \delta_{ml} \quad \forall m, l = 1, \ldots, 5, \quad \mathbf{r}_l \in \mathbf{ker}\,(\varepsilon)$.

Such an edge is called a fully primal edge.

We will sometimes write $f_m^{\mathcal{E}^{ik}}$ instead of f_m. We recall that the rigid body modes $\mathbf{r}_1, \ldots, \mathbf{r}_6$, restricted to an edge provide only five linearly independent vectors, since one rotation is always linearly dependent of other rigid body modes. In our arguments, we will assume that we have used an appropriate change of coordinates such that the edge under consideration coincides with the x_1-axis; the special rotation is then $\mathbf{r}_6$. As an example of functionals f_m, as required in Definition 2, we can use appropriate linear combinations of certain edge averages and first order moments of the components of the displacement given by

$$g_n(\mathbf{w}^{(i)}) := \frac{(\mathbf{w}^{(i)}, \mathbf{r}_n)_{L_2(\mathcal{E}^{ik})}}{(\mathbf{r}_n, \mathbf{r}_n)_{L_2(\mathcal{E}^{ik})}}, \quad n \in \{1, \ldots, 5\}. \tag{25}$$

Using a Cauchy-Schwarz inequality, we obtain

$$|g_n(\mathbf{w}^{(i)})|^2 \le \frac{\|\mathbf{w}^{(i)}\|^2_{L_2(\mathcal{E}^{ik})}}{\|\mathbf{r}_n\|^2_{L_2(\mathcal{E}^{ik})}}.$$

We can now proceed exactly as in the case of the fully primal faces and construct the functionals f_m in terms of a dual basis of $(\mathbf{ker}\,(\varepsilon))'$. Then, the second requirement of Definition 2 is again immediately satisfied by construction and the first requirement follows again from Lemma 11.

We do not have to require that every face and edge be fully primal but we need the concept of an acceptable face path for those that are not.

Definition 3 (Acceptable face path). *Consider a pair of subdomains (Ω_i, Ω_k) which have a face or an edge in common. An acceptable face path* $\{\Omega_i, \Omega_{j_1}, \ldots, \Omega_{j_n}, \Omega_k\}$

for this pair is a path from Ω_i to Ω_k, via a uniformly bounded number of other subdomains $\Omega_{j_q}, q = 1, \ldots, n$, such that the coefficients G_{j_q} of Ω_{j_q} satisfy the condition

$$TOL * G_{j_q} \geq \min(G_i, G_k) \quad q = 1, \ldots, n, \tag{26}$$

for some tolerance TOL. The path can pass from one subdomain to another only through a fully primal face; cf. Figure 1.

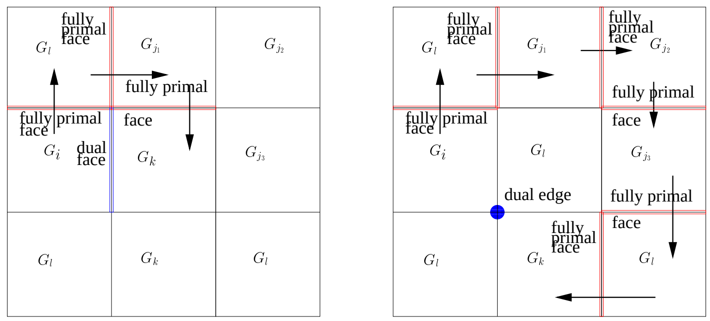

Fig. 1. *Examples of acceptable face paths (planar cut): dual face (left) and dual edge (right).*

We note that any fully primal face has a trivial acceptable face path. An edge should either be fully primal or have an acceptable face path.

Finally, we have to consider vertices, where we need to control only translational rigid body modes. A vertex is primal if the three displacement components are continuous. These variables are then global and this is reflected in the subassembly of the stiffness matrix of the preconditioner. We do not have to make a vertex primal if for every pair of subregions, which have the vertex in common, there is an acceptable vertex path.

Definition 4 (Acceptable vertex path). *Consider a pair of subdomains (Ω_i, Ω_k), which have a vertex in common. An acceptable vertex path $\{\Omega_i, \Omega_{j_1}, \ldots, \Omega_{j_n}, \Omega_k\}$ from Ω_i to Ω_k, is a path via a uniformly bounded number of other subdomains $\Omega_{j_q}, q = 1, \ldots, n$, such that the coefficients G_{j_q} of Ω_{j_q} satisfy the condition*

$$TOL * G_{j_q} \geq \frac{h_i}{H_i} \min(G_i, G_k) \quad q = 1, \ldots, n, \tag{27}$$

for some tolerance TOL. We can only pass from one subdomain to another through a fully primal face.

We will assume that for each pair (Ω_i, Ω_k), which has a face, an edge or a vertex, in common, there either exists an acceptable path as in Definitions 3 and 4, respectively, with a modest tolerance TOL and that the path does not exceed a prescribed length, or that the face or edge are fully primal or the vertex primal. (In Subsection 8.4, we will look in more detail at the consequences of long paths.) If TOL becomes too large for a certain face, edge, or vertex or if the length of the acceptable path exceeds a given uniform bound, we should make the face or edge fully primal, or the vertex primal; cf. Figure 2 for an example where certain vertices should be made primal.

We note that the bounds for the primal constraint functionals in Definitions 1 and 2 will allow us to prove almost uniform bounds for the condition number of our algorithms. If point constraints were to replace the edge constraints, this would not be possible. We note that while we will work with functionals which are not

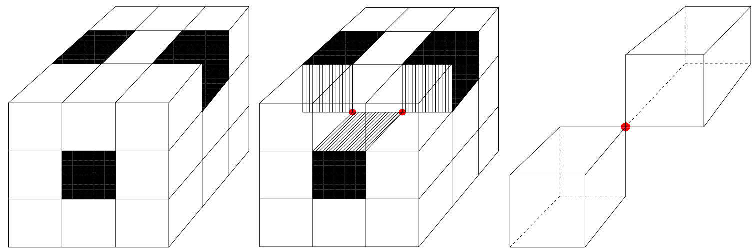

FIG. 2. *Example of a decomposition where no acceptable vertex path exists for the vertices which connect the black subdomains, which have much larger coefficients than those of the white. These vertices should be made primal.*

uniformly bounded, the growth of these bounds is quite modest when H/h grows. These growth factors will appear in the main theorem as is customary for many domain decomposition methods. We also note that the logarithmic factors cannot be eliminated if we wish to obtain a result which is uniform with respect to arbitrary variations of the Lamé parameters.

Finally, we define the spaces $\widehat{\mathbf{W}}_\Pi$ and $\widetilde{\mathbf{W}}_\Delta = \bigoplus_{i=1}^N \widetilde{\mathbf{W}}_\Delta^{(i)}$. For the definition of these spaces, we will use certain standard scalar finite element cutoff functions $\theta_{\mathcal{F}^{ij}}, \theta_{\mathcal{E}^{ik}}$, and $\theta_{\mathcal{V}^{il}}$. The first two equal one on $\mathcal{F}^{ij}$ and $\mathcal{E}^{ik}$, respectively, and vanish elsewhere on Γ_h; the cutoff function $\theta_{\mathcal{V}^{il}}$ equals one at the vertex and vanishes elsewhere on the interface. Additionally, for an edge, we denote by $m_{\mathcal{E}^{ik}}$ a linear function which equals 1 at one end of the edge and -1 at the other. The first space, $\widehat{\mathbf{W}}_\Pi$, is spanned component by component by the nodal finite element functions $\theta_{\mathcal{V}^{il}}$ which are associated with vertices $\mathcal{V}^{il}$ that have been chosen to be primal, by the cutoff functions $\theta_{\mathcal{E}^{ik}}$ for the primal edges, and for fully primal edges by $\theta_{\mathcal{E}^{ik}}$ and $I^h(m_{\mathcal{E}^{ik}}\theta_{\mathcal{E}^{ik}})$. Each such primal constraint is associated with a basis element of $\widehat{\mathbf{W}}_\Pi$; all these functions are continuous across the interface Γ. For each subdomain Ω_i, we then define a subspace $\widetilde{\mathbf{W}}_\Delta^{(i)}$ by those functions in $\mathbf{W}^{(i)}$ which vanish at the primal variables, i.e., these functions vanish at primal vertices and have zero averages over primal edges and additionally certain zero first order moments over fully primal edges. More details will be provided especially in Section 8.

6. Linear algebra aspects of the FETI-DP method. In this section, we first introduce matrix representations of the operators used in the description of our FETI-DP algorithm and given in Section 4. We then describe two ways of implementing these algorithms.

The matrix representing the jump operator B_Δ is constructed from $\{0, 1, -1\}$, in such a way that the values of the solution $\mathbf{u}_\Delta$, associated with more than one subdomain, coincide when $B_\Delta \mathbf{u}_\Delta = 0$. These constraints are very simple and just express that the nodal values coincide across the interface; in comparison with the one-level FETI method, cf. [13], we can drop some of the constraints, in particular those associated with the primal vertices. However, we will otherwise use all possible constraints and thus work with a fully redundant set of Lagrange multipliers as in [13, Section 5]; cf. also [23]. Thus, for an edge node common to four subdomains, we will use six constraints rather than choosing just three. To define the FETI-DP Dirichlet preconditioner, we also need to introduce a matrix representation of the scaled jump operator $B_{D,\Delta}$; this is done by scaling the contributions of B_Δ from individual subdomains. Additionally, we add a zero column to $B_{D,\Delta}$ for each primal vertex variable.

Let us now consider the matrix representation of our preconditioner M^{-1}. We need the matrix form of S_ε; this is a Schur complement matrix, which is obtained from the block diagonal matrix K, (8), by eliminating the interior variables. The associated block–diagonal matrix is given by

$$S_\varepsilon := diag_{i=1}^N(S_\varepsilon^{(i)}).$$

Each local stiffness matrix can be written as

$$K^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{\Gamma I}^{(i)T} \\ K_{\Gamma I}^{(i)} & K_{\Gamma\Gamma}^{(i)} \end{bmatrix}.$$

Then, each local Schur complement matrix $S_\varepsilon^{(i)}$ can be written as

$$S_\varepsilon^{(i)} = K_{\Gamma\Gamma}^{(i)} - K_{\Gamma I}^{(i)}(K_{II}^{(i)})^{-1}K_{\Gamma I}^{(i)T}.$$

Thus, we can compute S_ε times a vector $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$ by solving local Dirichlet problems and forming some sparse matrix-vector products. Our preconditioner is then given in matrix form by

$$M^{-1} = PB_{D,\Delta}S_\varepsilon B_{D,\Delta}^T P^T.$$

Finally, we have to consider the system matrix $F = B_\Delta \widetilde{S}_\varepsilon^{-1} B_\Delta^T$. We have to describe how the edge (and face) constraints can be implemented and how $\widetilde{S}_\varepsilon^{-1}$ times a vector can be computed efficiently. There are two different approaches to the edge and face constraints, one using optional Lagrange multipliers, which will form a part of the global, coarse problem, and the other which uses a change of basis; cf. Subsections 6.1 and 6.2, respectively. The second approach generally leads to smaller and computationally more efficient coarse problems; with this approach, the invertibility of the local problems and the positive definiteness of the entire problem can be guaranteed without any vertex constraints. In fact, vertex constraints are only needed for problems with very challenging distributions of the material coefficients. We will outline two different ways of implementing the change of basis; we can either explicitly carry out the basis transformation on the primal and fully primal edges for both the primal and dual displacement variables, or one can apply the transformation explicitly only for the primal displacement variables and use local Lagrange multipliers to enforce zero edge averages and first order moments for the dual displacements. The latter approach has the advantage of retaining more of the original sparsity of the stiffness matrices.

6.1. An implementation using global optional Lagrange multipliers. We first briefly review the approach taken by Farhat, Lesoinne, and Pierson in [9]. They assume that a sufficient number of vertices have been chosen as primal variables such that the stiffness matrix which results from K by partial assembly at those vertices is invertible even without any additional primal constraints. In two dimensions, such a set of vertex constraints is sufficient to obtain a fast and scalable algorithm but in three dimensions, to be competitive, we have to choose a primal space, which also ensures that certain face and/or edge averages and first order moments are the same across the interface. This approach can be implemented by introducing additional optional Lagrange multipliers originating from constraints of the form

$$Q_\Delta B_\Delta \mathbf{u}_\Delta = \sum_{i=1}^{N} Q_\Delta B_\Delta^{(i)} \mathbf{u}_\Delta^{(i)} = 0. \tag{28}$$

Here, Q_Δ is a rectangular matrix which has as many columns as there are Lagrange multipliers. It has one row for each primal constraint, which is not related to a vertex and the matrix Q_Δ is constructed such that (28) guarantees that certain linear combinations of the rows of $B_\Delta \mathbf{u}_\Delta$ vanish. Thus, these linear combinations are directly related to the constraints of the primal edges and faces; (28) forces appropriate edge averages and first order moments at fully primal edges or faces to have common values across the interface. We note that this approach could also be used for common face averages at selected faces across the interface; here we will work exclusively with edge averages and moments.

Let us now order the unknowns such that the interior and dual variables come first, grouped together in blocks by the subdomains and denoted with the subscript r, and that the primal vertices, with the subscript c, are ordered last. We note that the matrix $\widetilde{K}$ is partially assembled with respect to the primal vertices. Thus, we have

$$\widetilde{K} := \begin{bmatrix} K_{rr} & \widetilde{K}_{cr}^T & Q_r^T \\ \widetilde{K}_{cr} & \widetilde{K}_{cc} & O \\ Q_r & O & O \end{bmatrix},$$

where

$$K_{rr} := \begin{bmatrix} K_{rr}^{(1)} & & O \\ & \ddots & \\ O & & K_{rr}^{(N)} \end{bmatrix}, K_{rr}^{(i)} := \begin{bmatrix} K_{II}^{(i)} & K_{\Delta I}^{(i)T} \\ K_{\Delta I}^{(i)} & K_{\Delta\Delta}^{(i)} \end{bmatrix}, \widetilde{K}_{cr}^T := \begin{bmatrix} K_{cr}^{(1)T} R_c^{(1)T} \\ \vdots \\ K_{cr}^{(N)T} R_c^{(N)T} \end{bmatrix},$$

$$\widetilde{K}_{cc} = \sum_{i=1}^{N} R_c^{(i)} K_{cc}^{(i)} {R_c^{(i)}}^T, \quad Q_r := [Q_r^{(1)} \dots Q_r^{(N)}], \text{ and } Q_r^{(i)} := [O \;\; Q_\Delta B_\Delta^{(i)}].$$

Here, we denote by $R_c^{(i)}$ the matrix which performs the partial assembly at the relevant primal vertices and $\widetilde{K}_{cc}$ is the submatrix which is assembled at the primal vertices. The resulting leading two by two block of $\widetilde{K}$ is thus non singular. Using the notation $B_r := [B_r^{(1)} \dots B_r^{(N)}]$ with $B_r^{(i)} := [O \;\; B_\Delta^{(i)}]$, for a matrix with same structure as Q_r and introducing Lagrange multipliers $\boldsymbol{\lambda} \in \mathbf{range}\,(B_r)$, we can reformulate the original finite element problem as follows

$$\begin{bmatrix} K_{rr} & \widetilde{K}_{cr}^T & Q_r^T & B_r^T \\ \widetilde{K}_{cr} & \widetilde{K}_{cc} & O & O \\ Q_r & O & O & O \\ B_r & O & O & O \end{bmatrix} \begin{bmatrix} \mathbf{u}_r \\ \mathbf{u}_c \\ \boldsymbol{\mu} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_r \\ \mathbf{f}_c \\ 0 \\ 0 \end{bmatrix}.$$

We can now derive the unpreconditioned FETI-DP linear system by eliminating the variables $\mathbf{u}_r, \mathbf{u}_c$, and $\boldsymbol{\mu}$ to obtain $F\boldsymbol{\lambda} = \mathbf{d}$. In order to exploit the sparsity of $\widetilde{K}$, the Schur complement $\widetilde{S}_\varepsilon$ is never built explicitly. In fact, we only need to be able to compute the action of $\widetilde{S}_\varepsilon^{-1}$ on a vector $\mathbf{f}_\Delta$. This can be done in a computationally efficient way as described at the end of Section 6.2. We note that although the elimination of the interior and dual variables $\mathbf{u}_r$ leads to an indefinite system with respect to the primal variables $\mathbf{u}_c$ and the optional global Lagrange multipliers $\boldsymbol{\mu}$, it can still be solved without pivoting for stability. Instead, we can symmetrically reorder a large leading principal minor of our matrices so as to maintain sparsity. Since the argument is the same as for a variant described in the next section, we refer to the discussion at the end of Section 6.2.

6.2. An implementation using a change of basis. As a second approach, we present a method which uses a change of basis to force certain edge averages and first order moments to vanish. This change of basis will introduce the edge averages and moments as new primal variables and explicitly separate the dual from the primal variables. As a consequence, this will allow us to write our system matrix in a block structured form with respect to the interior, dual, and primal variables. Two variants are discussed, one where the transformation is carried out for the primal and selected dual displacement variables and a second one, where local Lagrange multipliers are used to enforce certain zero edge averages and moments instead of explicitly performing the change of basis for the associated dual displacement variables. Both approaches generally lead to smaller and computationally more efficient coarse problems. Such an implementation also works for face constraints instead of or in addition to edge constraints but since in this article we only consider edge based algorithms, we restrict ourselves to the case of edges only.

6.2.1. First approach. We first describe the approach using an explicit change of basis. As a result, the dual displacement vectors should have zero edge averages over primal edges and the selected displacement components and additionally zero first order edge moments over fully primal edges. In addition, we introduce these averages and moments as primal variables in $\widehat{\mathbf{W}}_\Pi$.

We first describe how the transformation matrix for such a change of basis can be built. We first consider the unknowns $\mathbf{u}_E$ on a fully primal edge $\mathcal{E}$. It is sufficient to describe the transformation of a single component of $\mathbf{u}_E = (u_{1,E}^T, u_{2,E}^T, u_{3,E}^T)^T$ subject to two constraints; we note that the edge average constraints are constructed for each of the three components whereas the constraints of first order moments are only used for two appropriately chosen components; cf. Section 5. For simplicity, we drop the subscripts indicating the component and consider only a scalar vector of unknowns $u_E = (u_1 \dots u_N)^T$. We define a transformation matrix T_E which performs the desired change of basis. In the new basis, we introduce the edge average $\bar{u}_E^a$ and the first order edge moment $\bar{u}_E^m$ as new variables. Additionally, the representation of the dual part of u_E in the new basis should have a zero edge average and a zero first order edge moment. Our transformation matrix T_E performs the change of basis from the new basis to the original nodal basis. If we denote the edge unknowns in the new basis by $\hat{u}_E$, we will have

$$u_E = T_E\, \hat{u}_E,$$

where

$$T_E = [t_1 \dots t_{N-2} t_{N-1} t_N].$$

We consider an edge component with two constraints and can define T_E in terms of second-order differences. The first $N-2$ vectors t_j are defined by first setting the j-th vector t_j to zero at all but the j-th and its next two components. We set the j-th entry to one and define the next two entries such that t_j has a zero average and a zero first order moment. The vector t_{N-1} is defined as being one at each mesh point of that edge and the last vector, t_N, is obtained by evaluating, at the mesh points, the linear function m which is orthogonal to t_{N-1} in the L_2-inner product; cf. the definition of $m_{\mathcal{E}^{ik}}$ at the end of Section 5.

A similar construction can be carried out for a primal edge. Then, only edge averages are introduced as new variables and the remaining new variables should have zero edge average. In this case the first $N-1$ columns of T_E are defined by setting all except for the j-th and $(j+1)$-th component to zero and making its average zero. The last column t_N is then obtained by setting all entries to one.

Such a transformation can be constructed separately for each component of $\mathbf{u}_E = (u_{1,E}^T, u_{2,E}^T, u_{3,E}^T)^T$ and for each edge with primal edge constraints. We denote the resulting transformation, which operates on all relevant components of $\mathbf{u}_E$ and all relevant edges, by $T_E^{(i)}$. The transformation for all variables of one subdomain Ω_i is then of the form

$$T^{(i)} = \begin{bmatrix} I & O & O \\ O & I & O \\ O & O & T_E^{(i)} \end{bmatrix},$$

where we assume that the variables are ordered interior variables first, interface variables not related to the (fully) primal edges second, and the variables on the (fully) primal edges last, i.e., a typical vector of nodal unknowns is of the form $[\mathbf{u}_I^{(i)T}, \mathbf{u}_{\overline{\Gamma}}^{(i)T}, \mathbf{u}_E^{(i)T}]^T$. We note that $T_E^{(i)}$ is a direct sum of the relevant transformation matrices associated with the primal and fully primal edges of that subdomain; $T_E^{(i)}$ is a block-diagonal matrix where each block represents the transformation for a component of a primal or fully primal edge.

Decomposing the subdomain stiffness matrices $K^{(i)}$ in the same manner, we obtain

$$K^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{I\overline{\Gamma}}^{(i)} & K_{IE}^{(i)} \\ K_{\overline{\Gamma}I}^{(i)} & K_{\overline{\Gamma}\overline{\Gamma}}^{(i)} & K_{\overline{\Gamma}E}^{(i)} \\ K_{EI}^{(i)} & K_{E\overline{\Gamma}}^{(i)} & K_{EE}^{(i)} \end{bmatrix}$$

Using the transformation $\mathbf{u}^{(i)} = T^{(i)}\widehat{\mathbf{u}}^{(i)}$, we obtain

$$T^{(i)T}K^{(i)}T^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{I\overline{\Gamma}}^{(i)} & K_{IE}^{(i)}T_E^{(i)} \\ K_{\overline{\Gamma}I}^{(i)} & K_{\overline{\Gamma}\overline{\Gamma}}^{(i)} & K_{\overline{\Gamma}E}^{(i)}T_E^{(i)} \\ T_E^{(i)T}K_{EI}^{(i)} & T_E^{(i)T}K_{E\overline{\Gamma}}^{(i)} & T_E^{(i)T}K_{EE}^{(i)}T_E^{(i)} \end{bmatrix}$$

The averages and moments are now new primal variables. We note that there might be additional primal variables, e.g., selected primal vertices. The primal variables belonging to Ω_i are denoted by $\hat{\mathbf{u}}_\Pi^{(i)}$ and the remaining, dual displacement variables by $\hat{\mathbf{u}}_\Delta^{(i)}$. By construction, the new dual displacement variables $\widehat{\mathbf{u}}_\Delta^{(i)}$ satisfy the zero edge average and moment properties. Using this decomposition of the unknowns into interior, dual, and primal displacement variables, the transformation matrix $T_E^{(i)}$ can be written as $[T_{\Delta_E}^{(i)}\ T_{\Pi_E}^{(i)}]$. Here, the indices Δ_E and Π_E indicate the dual and primal displacement variables associated with the primal edges constraints. Using this notation, we obtain

$$T^{(i)T}K^{(i)}T^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{I\overline{\Gamma}}^{(i)} & K_{I\Delta_E}^{(i)}T_{\Delta_E}^{(i)} & K_{I\Pi_E}^{(i)}T_{\Pi_E}^{(i)} \\ K_{\overline{\Gamma}I}^{(i)} & K_{\overline{\Gamma}\overline{\Gamma}}^{(i)} & K_{\overline{\Gamma}\Delta_E}^{(i)}T_{\Delta_E}^{(i)} & K_{\overline{\Gamma}\Pi_E}^{(i)}T_{\Pi_E}^{(i)} \\ T_{\Delta_E}^{(i)T}K_{\Delta_E I}^{(i)} & T_{\Delta_E}^{(i)T}K_{\Delta_E\overline{\Gamma}}^{(i)} & T_{\Delta_E}^{(i)T}K_{\Delta_E\Delta_E}^{(i)}T_{\Delta_E}^{(i)} & T_{\Delta_E}^{(i)T}K_{\Delta_E\Pi_E}^{(i)}T_{\Pi_E}^{(i)} \\ T_{\Pi_E}^{(i)T}K_{\Pi_E I}^{(i)} & T_{\Pi_E}^{(i)T}K_{\Pi_E\overline{\Gamma}}^{(i)} & T_{\Pi_E}^{(i)T}K_{\Pi_E\Delta_E}^{(i)}T_{\Delta_E}^{(i)} & T_{\Pi_E}^{(i)T}K_{\Pi_E\Pi_E}^{(i)}T_{\Pi_E}^{(i)} \end{bmatrix}.$$

If we denote the primal vertices by a subscript Π_V and the remaining dual displacement variables by a subscript Δ, we can then write $\mathbf{u}_{\overline{\Gamma}}^{(i)} = [\mathbf{u}_{\Delta}^{(i)T}\ \mathbf{u}_{\Pi_V}^{(i)T}]^T$. Using this splitting for the local stiffness matrices $K^{(i)}$ accordingly, ordering the primal variables $\mathbf{u}_{\Pi_V}^{(i)}$ and $\mathbf{u}_{\Pi_E}^{(i)}$ last, and combining them as primal variables $\mathbf{u}_{\Pi}^{(i)} = [\mathbf{u}_{\Pi_V}^{(i)T}, \mathbf{u}_{\Pi_E}^{(i)T}]^T$, we obtain

$$T^{(i)T}K^{(i)}T^{(i)} = \begin{bmatrix} K_{II}^{(i)} & \overline{K}_{\Delta I}^{(i)T} & \overline{K}_{\Pi I}^{(i)T} \\ \overline{K}_{\Delta I}^{(i)} & \overline{K}_{\Delta\Delta}^{(i)} & \overline{K}_{\Pi\Delta}^{(i)T} \\ \overline{K}_{\Pi I}^{(i)} & \overline{K}_{\Pi\Delta}^{(i)} & \overline{K}_{\Pi\Pi}^{(i)} \end{bmatrix}.$$

Here, we denote the transformed matrices by an overline. If we now assemble the primal contributions of each transformed $K^{(i)}$ and order the primal variables last, we obtain

$$\widetilde{K} := \begin{bmatrix} K_{II}^{(1)} & \overline{K}_{I\Delta}^{(1)} & & & & \widetilde{K}_{\Pi I}^{(1)T} \\ \overline{K}_{\Delta I}^{(1)} & \overline{K}_{\Delta\Delta}^{(1)} & & & & \widetilde{K}_{\Pi\Delta}^{(1)T} \\ & & \ddots & & & \vdots \\ & & & K_{II}^{(N)} & \overline{K}_{I\Delta}^{(N)} & \widetilde{K}_{\Pi I}^{(N)T} \\ & & & \overline{K}_{\Delta I}^{(N)} & \overline{K}_{\Delta\Delta}^{(N)} & \widetilde{K}_{\Pi\Delta}^{(N)T} \\ \widetilde{K}_{\Pi I}^{(1)} & \widetilde{K}_{\Pi\Delta}^{(1)} & \cdots & \widetilde{K}_{\Pi I}^{(N)} & \widetilde{K}_{\Pi\Delta}^{(N)} & \widetilde{K}_{\Pi\Pi} \end{bmatrix} =: \begin{bmatrix} \overline{K}_{BB} & \widetilde{K}_{\Pi B}^T \\ \widetilde{K}_{\Pi B} & \widetilde{K}_{\Pi\Pi} \end{bmatrix}.$$

To compute $\overline{\mathbf{u}}_{\Delta} = \widetilde{S}_{\varepsilon}^{-1}\overline{\mathbf{f}}_{\Delta}$, we solve the linear system

$$\widetilde{K}\mathbf{u} = \widehat{\mathbf{f}}, \tag{29}$$

with

$$\mathbf{u} := \begin{bmatrix} \mathbf{u}_B \\ \widetilde{\mathbf{u}}_{\Pi} \end{bmatrix} := \begin{bmatrix} \mathbf{u}_I^{(1)} \\ \overline{\mathbf{u}}_{\Delta}^{(1)} \\ \vdots \\ \mathbf{u}_I^{(N)} \\ \overline{\mathbf{u}}_{\Delta}^{(N)} \\ \widetilde{\mathbf{u}}_{\Pi} \end{bmatrix} \quad \text{and} \quad \widehat{\mathbf{f}} := \begin{bmatrix} \mathbf{f}_B \\ \mathbf{0} \end{bmatrix} := \begin{bmatrix} \mathbf{f}_I^{(1)} \\ \overline{\mathbf{f}}_{\Delta}^{(1)} \\ \vdots \\ \mathbf{f}_I^{(N)} \\ \overline{\mathbf{f}}_{\Delta}^{(N)} \\ \mathbf{0} \end{bmatrix}.$$

Elimination of the interior and dual displacement variables yields a Schur complement $\widetilde{S}_{\Pi}$. We note that this elimination can be carried out in parallel across the subdomains since the related matrix is block-diagonal. We have,

$$\widetilde{S}_{\Pi} := \widetilde{K}_{\Pi\Pi} - \widetilde{K}_{\Pi B}\overline{K}_{BB}^{-1}\widetilde{K}_{\Pi B}^T.$$

The Schur complement $\widetilde{S}_\Pi$ represents the global part of our preconditioner; its size equals the number of primal variables. Since this number typically is not very large, $\widetilde{S}_\Pi$ is explicitly built and factored. We now have all the ingredients for computing the solution of the linear system (29) which, by one step of block Gauss elimination, transforms into

$$\begin{bmatrix} \overline{K}_{BB} & \widetilde{K}_{\Pi B}^T \\ O & \widetilde{S}_\Pi \end{bmatrix} \begin{bmatrix} \mathbf{u}_B \\ \tilde{\mathbf{u}}_\Pi \end{bmatrix} = \begin{bmatrix} \mathbf{f}_B \\ -\widetilde{K}_{\Pi B}\overline{K}_{BB}^{-1}\mathbf{f}_B \end{bmatrix}.$$

6.2.2. Second approach. Although our transformation only affects the sparsity of our stiffness matrices relatively slightly, we will now describe an alternative way of enforcing the condition of zero averages and moments by local Lagrange multipliers. We note that the block-diagonal matrix $\overline{K}_{BB}$ will not be as sparse as before the change of variables. We will not explicitly build the matrix $\overline{K}_{BB}$ but we will instead enforce the average constraints with additional, local Lagrange multipliers $\boldsymbol{\mu}^{(i)}$. To derive this local system, let us consider the bilinear form associated with a local block from $\overline{K}_{BB}$. We have

$$\begin{aligned}
\mathbf{u}_B^{(i)T}\overline{K}_{BB}^{(i)}\mathbf{u}_B^{(i)} &= \begin{bmatrix} \mathbf{u}_I^{(i)} \\ \overline{\mathbf{u}}_\Delta^{(i)} \end{bmatrix}^T \begin{bmatrix} K_{II}^{(i)} & \overline{K}_{\Delta I}^{(i)T} \\ \overline{K}_{\Delta I}^{(i)} & \overline{K}_{\Delta\Delta}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(i)} \\ \overline{\mathbf{u}}_\Delta^{(i)} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{u}_I^{(i)} \\ \overline{\mathbf{u}}_\Delta^{(i)} \\ \mathbf{0} \end{bmatrix}^T \begin{bmatrix} K_{II}^{(i)} & \overline{K}_{\Delta I}^{(i)T} & \overline{K}_{\Pi I}^{(i)T} \\ \overline{K}_{\Delta I}^{(i)} & \overline{K}_{\Delta\Delta}^{(i)} & \overline{K}_{\Pi\Delta}^{(i)T} \\ \overline{K}_{\Pi I}^{(i)} & \overline{K}_{\Pi\Delta}^{(i)} & \overline{K}_{\Pi\Pi}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(i)} \\ \overline{\mathbf{u}}_\Delta^{(i)} \\ \mathbf{0} \end{bmatrix} \\
&= [\mathbf{u}_I^{(i)T}\overline{\mathbf{u}}_\Delta^{(i)T}\mathbf{0}^T]\ T^{(i)T}K^{(i)}T^{(i)}\ [\mathbf{u}_I^{(i)T}\overline{\mathbf{u}}_\Delta^{(i)T}\mathbf{0}^T]^T.
\end{aligned}$$

We first consider the case when there are no primal vertices and the only primal variables are edge averages and moments. Clearly, the last bilinear form on the right hand side minimizes the same energy as

$$\mathbf{u}^{(i)T}K^{(i)}\mathbf{u}^{(i)}$$

under the constraint $Q^{(i)}\mathbf{u}^{(i)} = 0$, where the local constraint matrices $Q^{(i)}$ force the edge averages and moments over the primal edges to vanish. The $Q^{(i)}$ can be derived in the same fashion as in Section 6.1; the only difference is that they are now defined locally.

If we also have primal vertices, we have to change $K^{(i)}$ slightly such that the homogeneous Dirichlet boundary conditions at the primal vertices are built in. This can be done either by setting the corresponding columns and rows to zero except for the diagonal elements, which are set to one, or these variables are eliminated and all further computations are carried out with the reduced matrix. In the following, we will always tacitly assume, without changing the notation, that one of these transformations has been carried out if we have primal vertices.

Thus, every time we have to solve a system with a local block from $\overline{K}_{BB}$, we instead solve a system

$$\begin{bmatrix} K_{II}^{(i)} & K_{I\Gamma}^{(i)} & O \\ K_{\Gamma I}^{(i)} & K_{\Gamma\Gamma}^{(i)} & {Q^{(i)}}^T \\ O & Q^{(i)} & O \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(i)} \\ \mathbf{u}_\Gamma^{(i)} \\ \boldsymbol{\mu}^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{f}_\Gamma^{(i)} \\ 0 \end{bmatrix}. \tag{30}$$

We denote the matrix on the left hand side by $K_Q^{(i)}$. We note that each $K_Q^{(i)}$ is invertible but that its leading two by two block is not (unless we have sufficiently many primal vertices). We therefore choose at least six auxiliary degrees of freedom for each subdomain if there are no primal vertices or otherwise fewer, according to the number of primal vertices. We order these auxiliary variables last in the leading two by two block of $K_Q^{(i)}$. Any such auxiliary variable can be associated with the interface or the interior of the subdomain. If the auxiliary variables are distributed between interior and boundary variables, some pivoting on small subsystems might be necessary. We indicate the auxiliary variables with an index A and the remaining interior variables with an index I_A. We now only discuss the case when all auxiliary variables are exclusively chosen among the interior variables, since in this case, no pivoting is necessary to maintain stability; we are freee to pivot for sparsity in the upper two by two block in (31). Denoting by P_A an appropriate permutation matrix which exchanges columns corresponding to the local Lagrange multipliers and the auxiliary variables, we have the following representation of $K_Q^{(i)}$:

$$(31)\, P_A^T K_Q^{(i)} P_A = \left[\begin{array}{cc|cc} K_{I_A I_A}^{(i)} & K_{I_A \Gamma}^{(i)} & O & K_{I_A A}^{(i)} \\ K_{I_A \Gamma}^{(i)T} & K_{\Gamma\Gamma}^{(i)} & Q_\Gamma^{(i)T} & K_{\Gamma A}^{(i)} \\ --- & --- & --- & --- \\ O & Q_\Gamma^{(i)} & O & O \\ K_{I_A A}^{(i)T} & K_{\Gamma A}^{(i)T} & O & K_{AA}^{(i)} \end{array} \right] =: \begin{bmatrix} \breve{K}_{II}^{(i)} & \breve{K}_{IA}^{(i)} \\ \breve{K}_{IA}^{(i)T} & \breve{K}_{AA}^{(i)} \end{bmatrix}.$$

We now again carry out one step of block Gaussian elimination and obtain the symmetric, indefinite Schur complement

$$S_A^{(i)} := \breve{K}_{AA}^{(i)} - \breve{K}_{AI}^{(i)T} \left(\breve{K}_{II}^{(i)} \right)^{-1} \breve{K}_{IA}^{(i)}.$$

This matrix is invertible since it is a Schur complement of an invertible matrix. We also define additional Schur complements, which are only needed for theoretical purposes to show that pivoting is not needed to factor $S_A^{(i)}$,

$$\begin{aligned} S_{\Gamma\Gamma}^{(i)} &:= K_{\Gamma\Gamma}^{(i)} - K_{\Gamma I_A}^{(i)} (K_{I_A I_A}^{(i)})^{-1} K_{I_A \Gamma}^{(i)}, \\ S_{\Gamma A}^{(i)} &:= K_{\Gamma A}^{(i)} - K_{\Gamma I_A}^{(i)} (K_{I_A I_A}^{(i)})^{-1} K_{I_A A}^{(i)}, \\ S_{AA}^{(i)} &:= K_{AA}^{(i)} - K_{A I_A}^{(i)} (K_{I_A I_A}^{(i)})^{-1} K_{I_A A}^{(i)}. \end{aligned}$$

The two Schur complements $S_{\Gamma\Gamma}^{(i)}$ and $S_{AA}^{(i)}$ are both symmetric, positive definite since they are obtained from such matrices. By a direct computation, $S_A^{(i)}$ has the form

$$S_A^{(i)} = \begin{bmatrix} -Q_\Gamma^{(i)} (S_{\Gamma\Gamma}^{(i)})^{-1} Q_\Gamma^{(i)T} & -Q_\Gamma^{(i)} (S_{\Gamma\Gamma}^{(i)})^{-1} S_{\Gamma A}^{(i)} \\ -S_{\Gamma A}^{(i)T} (S_{\Gamma\Gamma}^{(i)})^{-1} Q_\Gamma^{(i)T} & S_{AA}^{(i)} - S_{A\Gamma}^{(i)} (S_{\Gamma\Gamma}^{(i)})^{-1} S_{A\Gamma}^{(i)} \end{bmatrix} =: \begin{bmatrix} -A & -B^T \\ -B & C \end{bmatrix}.$$

If we assume that $Q_\Gamma^{(i)}$ has full column rank, which means that we only use non-redundant local Lagrange multipliers, then the matrix A is symmetric, positive definite. The matrix C is symmetric, positive semidefinite since it is a Schur complement of a positive semidefinite stiffness matrix. Thus, $C + BA^{-1}B^T$ is symmetric positive semidefinite as well. But since it is also a Schur complement of the invertible matrix $S_A^{(i)}$, it is even positive definite.

We conclude that, although the Schur complement $S_A^{(i)}$ is indefinite, it can still be solved without pivoting if the elimination is carried out blockwise, eliminating the local Lagrange multipliers before the auxiliary displacement variables.

In order to obtain the best possible sparsity, we can also allow for an arbitrary symmetric permutation of the rows and columns associated with the indices I_A and Γ since such a permutation keeps the Schur complement $S_A^{(i)}$ the same. This can be easily seen by a direct computation where the permutation matrices cancel when the Schur complement is built for the permuted block matrix.

To finally decide if the transformation of basis should be carried out for the dual displacement variables or if local Lagrange multipliers should be used has to be studied further by extensive numerical experiments.

7. Some auxiliary lemmas. The purpose of this section is to provide some technical lemmas which are needed in our convergence analysis in Section 8. These results are borrowed from [5, 7, 6]; see also [24, Sections 4.4 and 4.6] for an exposition of this material. Here, we formulate them using trace spaces on the subdomain boundaries, i.e., $H^{1/2}(\partial\Omega_i)$ instead of the spaces $H^1(\Omega_i)$ and discrete harmonic extensions; given the well–known equivalence of the norms, nothing essentially new needs to be proven. In our proofs, we will work with the S–norm defined by $|u|_S^2 = \sum_{i=1}^N |u^{(i)}|_{S^{(i)}}^2$ and $|u^{(i)}|_{S^{(i)}}^2 = \langle S^{(i)}u^{(i)}, u^{(i)}\rangle$. A proof of the equivalence of the $S^{(i)}$- and the $H^{1/2}(\partial\Omega_i)$-semi–norms of elements of $W^{(i)}$ can be found already in [1] for the case of piecewise linear elements and two dimensions and the tools necessary to extend this result to more general finite elements are provided in [25]; in our case, we of course have to multiply $|u^{(i)}|_{H^{1/2}(\partial\Omega_i)}^2$ by the factor G_i.

We also recall that we can define the $H_{00}^{1/2}(\tilde\Gamma)-$norm, $\tilde\Gamma \subset \partial\Omega_i$, of an element of $W^{(i)}$ which is supported in $\tilde\Gamma$, as the $H^{1/2}(\partial\Omega_i)-$norm of the function extended by zero on $\partial\Omega_{i,h} \setminus \tilde\Gamma_h$.

The first lemma can, essentially, be found in Dryja, Smith, and Widlund [5, Lemma 4.4]; see also [24, Lemma 4.25].

Lemma 8. *Let $\theta_{\mathcal{F}^{ij}}$ be the finite element function that is equal to 1 at the nodal points on the face $\mathcal{F}^{ij}$, which is common to two subregions Ω_i and Ω_j, and that vanishes on $(\partial\Omega_{i,h} \cup \partial\Omega_{j,h}) \setminus \mathcal{F}_h^{ij}$. Furthermore, let ϕ be a linear function on Ω_i. Then,*

$$|I^h(\theta_{\mathcal{F}^{ij}}\phi)|_{H^{1/2}(\partial\Omega_i)}^2 \le C(1+\log(H_i/h_i))H_i\|\phi\|_{L^\infty(\mathcal{F}^{ij})}^2.$$

The same bounds also hold for the other subregion Ω_j.

The following result can, essentially, be found in Dryja, Smith, and Widlund [5, Lemma 4.5] in Dryja [4, Lemma 3] or in [24, Lemma 4.24].

Lemma 9. *Let $\theta_{\mathcal{F}^{ij}}$ be the function introduced in Lemma 8 and let I^h denote the interpolation operator onto the finite element space $\mathbf{W}^h(\Omega_i)$. Then, for all $u \in W^{(i)}$,*

$$\|I^h(\theta_{\mathcal{F}^{ij}}u)\|_{H_{00}^{1/2}(\mathcal{F}^{ij})}^2 \le C(1+\log(H_i/h_i))^2\big(|u|_{H^{1/2}(\mathcal{F}^{ij})}^2 + \frac{1}{H_i}\|u\|_{L_2(\mathcal{F}^{ij})}^2\big).$$

We will also need two additional results which are used to estimate the contributions to our bounds from the edges of Ω_i. For the next lemma, see Dryja, Smith, and Widlund [5, Lemma 4.7] or [24, Lemma 4.19].

Lemma 10. *Let $\theta_{\mathcal{E}^{ik}}$ be the cutoff function associated with the edge $\mathcal{E}^{ik}$. Then, for all $u \in W^{(i)}$,*

$$|I^h(\theta_{\mathcal{E}^{ik}}u)|_{H^{1/2}(\partial\Omega_i)}^2 \le C\|u\|_{L_2(\mathcal{E}^{ik})}^2.$$

This result follows by an elementary estimate of the energy norm of the zero extension of the boundary values and by noting that the harmonic extension has a smaller energy.

We will also need a Sobolev-type inequality for finite element functions, see Dryja and Widlund [6, Lemma 3.3] or Dryja [4, Lemma 1] or [24, Lemma 4.17].

LEMMA 11. *Let $\mathcal{E}^{ik}$ be any edge of Ω_i, which forms part of the boundary of a face $\mathcal{F}^{ij} \subset \partial\Omega_i$. Then, for all $u \in W^{(i)}$,*

$$\|u\|^2_{L_2(\mathcal{E}^{ik})} \leq C(1 + \log(H_i/h_i))\big(|u|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_i}\|u\|^2_{L_2(\mathcal{F}^{ij})}\big).$$

The next lemma can be found in Toselli and Widlund [24, Lemma 4.28].

LEMMA 12. *Let $\mathcal{V}^{il}$ be a vertex of a substructure Ω_i and let $u \in W^{(i)}$. Then,*

$$|u(\mathcal{V}^{il})\theta_{\mathcal{V}^{il}}|^2_{H^{1/2}(\partial\Omega_i)} \leq C\,(|u|^2_{H^{1/2}(\partial\Omega_i)} + 1/H_i\|u\|^2_{L_2(\partial\Omega_i)}).$$

8. Convergence analysis. As in [15], the two different Schur complements, $\widetilde{S}_\varepsilon$ and S_ε, introduced in Section 4, play an important role in the analysis of the dual–primal iterative algorithm. Both operate on the second subspace $\widetilde{\mathbf{W}}_\Delta$ and we also recall that $\widetilde{S}_\varepsilon$ represents a global problem while S_ε does not. For $\mathbf{u}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, we define the following seminorm, cf. (12),

$$|\mathbf{u}_\Delta|_{\widetilde{S}_\varepsilon} := \langle \widetilde{S}_\varepsilon \mathbf{u}_\Delta, \mathbf{u}_\Delta\rangle^{1/2}. \tag{32}$$

Let $\mathbf{V} := \mathbf{range}\,(B_\Delta)$ be the space of Lagrange multipliers. As in [13, Section 5], we introduce a projection

$$P_\Delta := B^T_{D,\Delta} B_\Delta.$$

A simple computation shows, see [13, Lemma 4.2], that P_Δ preserves the jump of any function $\mathbf{u}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, i.e.,

$$B_\Delta P_\Delta \mathbf{u}_\Delta = B_\Delta \mathbf{u}_\Delta \tag{33}$$

and we also have $P_\Delta \mathbf{u} = 0 \;\; \forall \mathbf{u} \in \widehat{\mathbf{W}}$.

Let $x \in \Gamma_h$ and let $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$. We borrow the following formula from [13, (4.4)]:

$$(P_\Delta \mathbf{w}_\Delta)^{(i)}(x) \;=\; \sum_{j \in \mathcal{N}_x} \delta_j^\dagger(\mathbf{w}_\Delta^{(i)}(x) - \mathbf{w}_\Delta^{(j)}(x)), x \in \partial\Omega_{i,h} \cap \Gamma_h. \tag{34}$$

Here, $\mathcal{N}_x$ is the set of indices of the subregions which have the node x on its boundary. We note that the coefficients in this expression are constant on the set of nodal points of each face and each edge of $\partial\Omega_i$, and that this formula is independent of the particular choice of B_Δ. We can now show that our preconditioner is invertible.

LEMMA 13. *The preconditioner M^{-1} is invertible whenever $\widetilde{S}_\varepsilon$ is.*

Proof. We first note that any null vector of S_ε is a piecewise rigid body mode. A nonsingular $\widetilde{S}_\varepsilon$ means that we have enough primal constraints across the interface Γ to rule out any nontrivial vector of this kind. These constraints can all be formulated as linear functionals operating on jumps, $\mathbf{w}^{(i)} - \mathbf{w}^{(j)}$, of $\mathbf{w} \in \widetilde{W}$. It is also easy to see that any null vector $\boldsymbol{\mu} = B_\Delta \mathbf{w}$ of M^{-1} corresponds to a null vector $P_\Delta \mathbf{w}$ of S_ε, which therefore must be a piecewise rigid body mode. The vector $P_\Delta \mathbf{w}$ must satisfy all the same constraints as $\mathbf{w} \in \widetilde{W}$ since $\mathbf{w} - P_\Delta \mathbf{w}$ is continuous and thus has no jumps across Γ. The set of constraints satisfied by $\mathbf{w} \in \widetilde{W}$ is therefore inherited by $P_\Delta \mathbf{w}$ and since this vector is a piecewise rigid body mode, it must vanish.

□

In our proof of Theorem 1, we will use representation formulas for F and M, which will allow us to carry out our analysis in the space of displacement variables. The representation formula for F is given in the next lemma; see also Klawonn, Widlund, and Dryja [15, p. 175] or Mandel and Tezaur [20]. In the proof of the next lemma, we will use the fact that $\widehat{\mathbf{W}}$ is the null space of B_Δ.

LEMMA 14. *For any $\boldsymbol{\lambda} \in \mathbf{V}$, we have*

$$\langle F\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \sup_{0 \neq \mathbf{v} \in \widetilde{\mathbf{W}}} \frac{\langle \boldsymbol{\lambda}, B_\Delta \mathbf{v} \rangle^2}{|\mathbf{v}|^2_{S_\varepsilon}}.$$

Proof. Using the definition of F, we find

$$\begin{aligned}
\langle F\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle &= \langle \widetilde{S}_\varepsilon^{-1} B_\Delta^T \boldsymbol{\lambda}, B_\Delta^T \boldsymbol{\lambda} \rangle \\
&= |\widetilde{S}_\varepsilon^{-1/2} B_\Delta^T \boldsymbol{\lambda}|^2 \\
&= \sup_{0 \neq \mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta} \frac{\langle \widetilde{S}_\varepsilon^{-1/2} B_\Delta^T \boldsymbol{\lambda}, \mathbf{w}_\Delta \rangle^2}{|\mathbf{w}_\Delta|^2} \\
&= \sup_{0 \neq \mathbf{v}_\Delta \in \widetilde{\mathbf{W}}_\Delta} \frac{\langle \boldsymbol{\lambda}, B_\Delta \mathbf{v}_\Delta \rangle^2}{|\mathbf{v}_\Delta|^2_{\widetilde{S}_\varepsilon}} \\
&= \sup_{0 \neq \mathbf{v}_\Delta \in \widetilde{\mathbf{W}}_\Delta} \frac{\langle \boldsymbol{\lambda}, B_\Delta \mathbf{v}_\Delta \rangle^2}{\inf_{\mathbf{v}_\Pi \in \widehat{\mathbf{W}}_\Pi} |\mathbf{v}_\Delta + \mathbf{v}_\Pi|^2_{S_\varepsilon}} \\
&= \sup_{0 \neq \mathbf{v} \in \widetilde{\mathbf{W}}} \frac{\langle \boldsymbol{\lambda}, B_\Delta \mathbf{v} \rangle^2}{|\mathbf{v}|^2_{S_\varepsilon}}
\end{aligned}$$

□

A similar formula holds for M; it only differs in the denominator from the one for F.

LEMMA 15. *For any $\boldsymbol{\lambda} \in \mathbf{V}$, we have*

$$\langle M\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \sup_{0 \neq \mathbf{v} \in \widetilde{\mathbf{W}}} \frac{\langle \boldsymbol{\lambda}, B_\Delta \mathbf{v} \rangle^2}{|P_\Delta \mathbf{v}|^2_{S_\varepsilon}}.$$

Proof. Using the definition of M^{-1} and of the projection P, see (23) and the following lines, and the fact that $P^T \boldsymbol{\nu} = P\boldsymbol{\nu} = \boldsymbol{\nu}$ for $\boldsymbol{\nu} \in \mathbf{V}$, we obtain

$$\begin{aligned}
\langle M\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle &= |M^{1/2} \boldsymbol{\lambda}|^2 \\
&= \sup_{\boldsymbol{\mu} \in \mathbf{V}} \frac{\langle M^{1/2} \boldsymbol{\lambda}, \boldsymbol{\mu} \rangle^2}{|\boldsymbol{\mu}|^2} \\
&= \sup_{\boldsymbol{\nu} \in \mathbf{V}} \frac{\langle \boldsymbol{\lambda}, \boldsymbol{\nu} \rangle^2}{\langle M^{-1} \boldsymbol{\nu}, \boldsymbol{\nu} \rangle}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\boldsymbol{\nu}\in\mathbf{V}} \frac{\langle \boldsymbol{\lambda}, \boldsymbol{\nu}\rangle^2}{\langle S_\varepsilon B_{D,\Delta}^T \boldsymbol{\nu}, B_{D,\Delta}^T \boldsymbol{\nu}\rangle} \\
&= \sup_{\mathbf{v}\in\widetilde{\mathbf{W}}} \frac{\langle \boldsymbol{\lambda}, B_\Delta \mathbf{v}\rangle^2}{\langle S_\varepsilon B_{D,\Delta}^T B_\Delta \mathbf{v}, B_{D,\Delta}^T B_\Delta \mathbf{v}\rangle}
\end{aligned}
$$

□

For a proof of the lower bound in our main theorem, we will use the following lemma:

LEMMA 16. *For any $\boldsymbol{\mu} \in \mathbf{V}$, there exists a $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$ such that $\boldsymbol{\mu} = B_\Delta \mathbf{w}_\Delta$ and $(I - P_\Delta)\mathbf{w}_\Delta \in \widehat{\mathbf{W}}_\Pi$. In addition, $\mathbf{z}_w = P_\Delta \mathbf{w}_\Delta \in \widetilde{\mathbf{W}}$ and $\boldsymbol{\mu} = B_\Delta \mathbf{z}_w$.*

Proof. Let μ be an arbitrary element in $\mathbf{V}$. Since $\mathbf{V} = \mathbf{range}\,(B_\Delta)$, there are many elements $\mathbf{u}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, such that $\boldsymbol{\mu} = B_\Delta \mathbf{u}_\Delta$. Given any such $\mathbf{u}_\Delta$, we write it as

$$
\mathbf{u}_\Delta = P_\Delta \mathbf{u}_\Delta + E_\Delta \mathbf{u}_\Delta,
$$

where $E_\Delta = I - P_\Delta$. While $E_\Delta \mathbf{u}_\Delta \in \widehat{\mathbf{W}}$, it is not necessarily an element in the subspace $\widehat{\mathbf{W}}_\Pi$. But since $\widehat{\mathbf{W}} \subset \mathbf{W}$ and using (9), we can always write it as the sum of its dual and primal components

$$
E_\Delta \mathbf{u}_\Delta = (E_\Delta \mathbf{u}_\Delta)_\Delta + (E_\Delta \mathbf{u}_\Delta)_\Pi,
$$

where $(E_\Delta \mathbf{u}_\Delta)_\Delta \in \widetilde{\mathbf{W}}_\Delta$ and $(E_\Delta \mathbf{u}_\Delta)_\Pi \in \widehat{\mathbf{W}}_\Pi$. We denote by $\widehat{\mathbf{w}}_\Pi$ the primal component $(E_\Delta \mathbf{u}_\Delta)_\Pi$ and we define

$$
\mathbf{w}_\Delta := P_\Delta \mathbf{u}_\Delta + \widehat{\mathbf{w}}_\Pi.
$$

The resulting element $\mathbf{w}_\Delta$ is in the dual subspace $\widetilde{\mathbf{W}}_\Delta$. This follows directly from

$$
\mathbf{w}_\Delta = P_\Delta \mathbf{u}_\Delta + \widehat{\mathbf{w}}_\Pi = \mathbf{u}_\Delta - (E_\Delta \mathbf{u}_\Delta)_\Delta
$$

and the fact that the right hand side, by construction, is in $\widetilde{\mathbf{W}}_\Delta$. Since $\widehat{\mathbf{w}}_\Pi$ is continuous across the interface, $B_\Delta \mathbf{w}_\Delta = \mu$ follows directly from (33). We can finally define $\mathbf{z}_w$ with the right properties as a sum of two elements in $\widetilde{\mathbf{W}}_\Delta$ and $\widehat{\mathbf{W}}_\Pi$, respectively:

$$
\mathbf{z}_w := \mathbf{w}_\Delta + (P_\Delta \mathbf{w}_\Delta - \mathbf{w}_\Delta) = P_\Delta \mathbf{w}_\Delta \in \widetilde{\mathbf{W}}.
$$

The proof is concluded by using (33).

□

We now require P_Δ to satisfy a stability condition which we will prove for different cases in the further course of the paper; see Subsections 8.1, 8.2, and 8.3.

CONDITION 1. *For all $\mathbf{w} \in \widetilde{\mathbf{W}}$, we have,*

$$
|P_\Delta \mathbf{w}|_{S_\varepsilon}^2 \leq C \, \max(1, TOL)\, (1 + \log(H/h))^2 |\mathbf{w}|_{S_\varepsilon}^2.
$$

Using Condition 1 and the three previous lemmas, we can now prove our condition number estimate.

THEOREM 1. *The condition number satisfies*

$$
\kappa(M^{-1}F) \leq C \, \max(1, TOL)\, (1 + \log(H/h))^2.
$$

Here, C is independent of h, H, γ, and the values of the G_i.

Proof. We have to estimate the smallest eigenvalue $\lambda_{min}(M^{-1}F)$ from below and the largest eigenvalue $\lambda_{max}(M^{-1}F)$ from above. We will show that

$$(35)\quad \langle M\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle \leq \langle F\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle \leq C\,\max(1,TOL)\,(1+\log(H/h))^2\langle M\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle \quad \forall \boldsymbol{\lambda}\in\mathbf{V}.$$

Lower bound: The lower bound follows by using Lemmas 14, 15, and 16: $\forall \boldsymbol{\lambda}\in\mathbf{V}$, we have

$$\langle M\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle = \sup_{\mathbf{w}\in\widetilde{\mathbf{W}}}\frac{\langle\boldsymbol{\lambda},B_\Delta\mathbf{w}\rangle^2}{|P_\Delta\mathbf{w}|^2_{S_\varepsilon}} = \sup_{\mathbf{w}\in\widetilde{\mathbf{W}}}\frac{\langle\boldsymbol{\lambda},B_\Delta\mathbf{z}_w\rangle^2}{|\mathbf{z}_w|^2_{S_\varepsilon}} \leq \sup_{\mathbf{z}\in\widetilde{\mathbf{W}}}\frac{\langle\boldsymbol{\lambda},B_\Delta\mathbf{z}\rangle^2}{|\mathbf{z}|^2_{S_\varepsilon}} = \langle F\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle.$$

Upper bound: Using Condition 1 and Lemmas 14 and 15, we obtain $\forall \boldsymbol{\lambda}\in\mathbf{V}$

$$\begin{aligned}\langle F\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle &= \sup_{0\neq\mathbf{w}\in\widetilde{\mathbf{W}}}\frac{\langle\boldsymbol{\lambda},B_\Delta\mathbf{w}\rangle^2}{|\mathbf{w}|^2_{S_\varepsilon}}\\ &\leq C\,\max(1,TOL)\,(1+\log(H/h))^2\sup_{0\neq\mathbf{w}\in\widetilde{\mathbf{W}}}\frac{\langle\boldsymbol{\lambda},B_\Delta\mathbf{w}\rangle^2}{|P_\Delta\mathbf{w}|^2_{S_\varepsilon}}\\ &= C\,\max(1,TOL)\,(1+\log(H/h))^2\langle M\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle.\end{aligned}$$

□

We will now establish Condition 1 successively for different cases.

8.1. First case. Let us first consider a decomposition of Ω, where no more than three subdomains are common to any edge and where each of the subdomains shares a face with each of the other two as in Figure 3. We further assume that all vertices

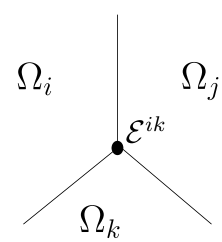

FIG. 3. *Planar cut of three subdomains meeting at an edge.*

are primal and that all faces are fully primal, cf. Definition 1. Thus, for each face $\mathcal{F}^{ij}$ which is shared by two subdomains Ω_i and Ω_j, we have six linear functionals $f_m(\cdot)$ which satisfy the two conditions of Definition 1 and have the property that $f_m(\mathbf{w}^{(i)}) = f_m(\mathbf{w}^{(j)})\quad \forall \mathbf{w}^{(i)}\in\widetilde{\mathbf{W}}^{(i)}, \mathbf{w}^{(j)}\in\widetilde{\mathbf{W}}^{(j)}$. As mentioned before, cf. the example after Definition 1, we can define our functionals f_i as properly chosen linear combinations of certain edge averages, over components of the displacement, of the form

$$g_m(\mathbf{w}^{(i)}) = \frac{\int_{\mathcal{E}} w_l^{(i)}dx}{\int_{\mathcal{E}} 1dx},$$

where $\mathcal{E}\subset\partial\mathcal{F}^{ij}$ are appropriately chosen edges. We note that for a square face we would have to work with at least three different edges to satisfy the second condition of Definition 1. For this case, we are able to prove Condition 1 with $TOL = 1$.

LEMMA 17. *For all* $\mathbf{w} \in \widetilde{\mathbf{W}}$, *we have,*

$$|P_\Delta \mathbf{w}|_{S_\varepsilon}^2 \leq C\,(1+\log(H/h))^2 |\mathbf{w}|_{S_\varepsilon}^2.$$

Proof. We consider an arbitrary $\mathbf{w} \in \widetilde{\mathbf{W}}$. Then, using formula (14), we see that it is sufficient to show that

$$|P_\Delta \mathbf{w}|_{S}^2 \leq C\,(1+\log(H/h))^2 |\mathbf{w}|_{S_\varepsilon}^2.$$

With $\mathbf{v}^{(i)} := (P_\Delta \mathbf{w})^{(i)}$, we have

$$|P_\Delta \mathbf{w}|_{S}^2 = \sum_{i=1}^{N} |\mathbf{v}^{(i)}|_{S^{(i)}}^2.$$

We cut each function $\mathbf{v}^{(i)}$ using the partition-of-unity functions $\theta_{\mathcal{F}^{ij}}, \theta_{\mathcal{E}^{ik}}$, and $\theta_{\mathcal{V}^{il}}$ and write it as a sum of terms which vanish at all the interface nodes outside individual faces, edges, and vertices, respectively. We note that the vertex terms vanish since all vertices are primal. Then, we obtain

$$\mathbf{v}^{(i)} = \sum_{\mathcal{F}^{ij} \subset \partial\Omega_i} I^h(\theta_{\mathcal{F}^{ij}} \mathbf{v}^{(i)}) + \sum_{\mathcal{E}^{ik} \subset \partial\Omega_i} I^h(\theta_{\mathcal{E}^{ik}} \mathbf{v}^{(i)}). \tag{36}$$

Face terms

We find that the face $\mathcal{F}^{ij}$ contributes

$$I^h(\theta_{\mathcal{F}^{ij}} \delta_j^\dagger (\mathbf{w}^{(i)} - \mathbf{w}^{(j)})).$$

This formula follows from (34) and we have to estimate the $H_{00}^{1/2}(\mathcal{F}^{ij})-$norm of this term. Since all faces are fully primal, we know that the functionals $f_m^{\mathcal{F}^{ij}}(\cdot) := f_m(\cdot)$ satisfy $f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)}) = f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(j)}), m = 1, \ldots, 6$, and $f_m^{\mathcal{F}^{ij}}(\mathbf{r}_n) = \delta_{mn}, m, n = 1, \ldots, 6$; cf. Definition 1. We have

$$\mathbf{w}^{(i)} - \mathbf{w}^{(j)} = \left(\mathbf{w}^{(i)} - \sum_{m=1}^{6} f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)}) r_m\right) - \left(\mathbf{w}^{(j)} - \sum_{m=1}^{6} f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(j)}) r_m\right). \tag{37}$$

Using the representation of an arbitrary rigid body mode $\mathbf{r}^{(i)} \in \mathbf{W}^{(i)}$, in terms of the basis $(\mathbf{r}_m)_{m=1,\ldots,6}$ of $\mathbf{ker}\,(\varepsilon)$, we easily obtain

$$\mathbf{r}^{(i)} = \sum_{m=1}^{6} f_m^{\mathcal{F}^{ij}}(\mathbf{r}) \mathbf{r}_m. \tag{38}$$

The first term on the right hand side of (37), can then be written as

$$\mathbf{w}^{(i)} - \textstyle\sum_{m=1}^{6} f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)}) \mathbf{r}_m \quad = \quad (\mathbf{w}^{(i)} - \mathbf{r}^{(i)}) - \textstyle\sum_{m=1}^{6} f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)} - \mathbf{r}^{(i)}) \mathbf{r}_m \tag{39}$$

for any rigid body mode $\mathbf{r}^{(i)} \in \mathbf{W}^{(i)}$. We can estimate the first term on the right hand side, using Lemmas 9 and 6, and find

$$\|I^h(\theta_{\mathcal{F}^{ij}}(\mathbf{w}^{(i)} - \mathbf{r}^{(i)}))\|_{H_{00}^{1/2}(\mathcal{F}^{ij})}^2$$

$$
\begin{aligned}
&\le \quad C(1+\log(H_i/h_i))^2\left(|\mathbf{w}^{(i)}-\mathbf{r}^{(i)}|^2_{H^{1/2}(\mathcal{F}^{ij})}+\frac{1}{H_i}\|\mathbf{w}^{(i)}-\mathbf{r}^{(i)}\|^2_{L_2(\mathcal{F}^{ij})}\right)\\
&\le \quad C(1+\log(H_i/h_i))^2\left(|\mathbf{w}^{(i)}-\mathbf{r}^{(i)}|^2_{E(\mathcal{F}^{ij})}+\frac{1}{H_i}\|\mathbf{w}^{(i)}-\mathbf{r}^{(i)}\|^2_{L_2(\mathcal{F}^{ij})}\right)\\
&\le \quad C(1+\log(H_i/h_i))^2\left(|\mathbf{w}^{(i)}|^2_{E(\mathcal{F}^{ij})}+\frac{1}{H_i}\|\mathbf{w}^{(i)}-\mathbf{r}^{(i)}\|^2_{L_2(\mathcal{F}^{ij})}\right)
\end{aligned}
$$

Next, we consider the second term of (39). We use two auxiliary estimates. By using Lemma 8, we see that

$$
\|I^h(\theta_{\mathcal{F}^{ij}}\mathbf{r}_k)\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}\le C\,H_i(1+\log(H_i/h_i)) \tag{40}
$$

and by using Definition 1 and Lemmas 9 and 6, we obtain as before

$$
|f_k^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})|^2 \quad \le \quad C\,H_i^{-1}\,(1+\log(H_i/h_i))\left(|\mathbf{w}^{(i)}|^2_{E(\mathcal{F}^{ij})}+\frac{1}{H_i}\|\mathbf{w}^{(i)}-\mathbf{r}^{(i)}\|^2_{L_2(\mathcal{F}^{ij})}\right).
$$

Thus, we have

$$
\begin{aligned}
&\|I^h(\theta_{\mathcal{F}^{ij}}(\sum_{m=1}^{6}f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})\mathbf{r}_m))\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}\\
\le \quad & C(1+\log(H_i/h_i))^2\left(|\mathbf{w}^{(i)}|^2_{E(\mathcal{F}^{ij})}+\frac{1}{H_i}\|\mathbf{w}^{(i)}-\mathbf{r}^{(i)}\|^2_{L_2(\mathcal{F}^{ij})}\right).
\end{aligned}
$$

Using (39) and the triangular inequality, we obtain the following estimate

$$
\begin{aligned}
&G_i\|I^h(\theta_{\mathcal{F}^{ij}}(\mathbf{w}^{(i)}-\sum_{m=1}^{6}f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)})\mathbf{r}_m))\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}\\
= \quad & G_i\|I^h(\theta_{\mathcal{F}^{ij}}((\mathbf{w}^{(i)}-\mathbf{r}^{(i)})-\sum_{m=1}^{6}f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})\mathbf{r}_m))\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}\\
\le \quad & 2G_i\|I^h(\theta_{\mathcal{F}^{ij}}(\mathbf{w}^{(i)}-\mathbf{r}^{(i)}))\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}+2G_i\|I^h(\theta_{\mathcal{F}^{ij}}(\sum_{m=1}^{6}f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})\mathbf{r}_m))\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}\\
\le \quad & C\,(1+\log(H_i/h_i))^2\,G_i\,\left(|\mathbf{w}^{(i)}|^2_{E(\mathcal{F}^{ij})}+\frac{1}{H_i}\|\mathbf{w}^{(i)}-\mathbf{r}^{(i)}\|^2_{L_2(\mathcal{F}^{ij})}\right).
\end{aligned}
$$

Since $\mathbf{r}^{(i)}\in\mathbf{W}^{(i)}$ is an arbitrary rigid body mode, we can take the infimum over all rigid body modes in $\mathbf{W}^{(i)}$ and obtain, by using Lemma 7,

$$
G_i\|I^h(\theta_{\mathcal{F}^{ij}}(\mathbf{w}^{(i)}-\sum_{m=1}^{6}f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)})\mathbf{r}_m))\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}\le C\,(1+\log(H_i/h_i))^2\,G_i\,|\mathbf{w}^{(i)}|^2_{E(\mathcal{F}^{ij})}.
$$

Analogously, we obtain by using the minimizing $\mathbf{r}^{(j)}\in\mathbf{W}^{(j)}$,

$$
G_j\|I^h(\theta_{\mathcal{F}^{ij}}(\mathbf{w}^{(j)}-\sum_{m=1}^{6}f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(j)})\mathbf{r}_m))\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})}\le C(1+\log(H_j/h_j))^2G_j\,|\mathbf{w}^{(j)}|^2_{E(\mathcal{F}^{ij})}.
$$

Using these two estimates in combination with (37) and the triangular inequality, we obtain

$$\begin{aligned}
& G_i\|\delta_j^\dagger I^h(\theta_{\mathcal{F}^{ij}}(\mathbf{w}^{(i)}-\mathbf{w}^{(j)}))\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})} \\
= \;& G_i\|\delta_j^\dagger I^h(\theta_{\mathcal{F}^{ij}}((\mathbf{w}^{(i)}-\sum_{k=1}^{6} f_k^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)})\mathbf{r}_k)-(\mathbf{w}^{(j)}-\sum_{k=1}^{6} f_k^{\mathcal{F}^{ij}}(\mathbf{w}^{(j)})\mathbf{r}_k))\|^2_{H^{1/2}_{00}(\mathcal{F}^{ij})} \\
\leq \;& C(1+\log(H_i/h_i))^2 G_i|\mathbf{w}^{(i)}|^2_{E(\mathcal{F}^{ij})} + C(1+\log(H_j/h_j))^2 G_j|\mathbf{w}^{(j)}|^2_{E(\mathcal{F}^{ij})}.
\end{aligned}$$

Edge terms

We now estimate the edge contributions. As in the scalar case, we have to estimate L_2-terms related to edges. By Lemma 10, we can estimate the contribution of the edges of Ω_i to the energy of $\mathbf{v}^{(i)}$ in terms of L_2-norms over the edges. These L_2-norms are then estimated by using Lemma 11.

Since we consider here the case with at most three subdomains common to a single edge, we can reduce the estimate of the edge contributions to face estimates. If three subdomains, e.g., Ω_i, Ω_j, and Ω_k, have an edge $\mathcal{E}^{ik}$ in common, cf. Figure 3, then, according to (34), there are two contributions to the estimate of the contribution of Ω_i to $|P_D\mathbf{w}|_S$, namely,

$$G_i\|\delta_j^\dagger(\mathbf{w}^{(i)}-\mathbf{w}^{(j)})\|^2_{L_2(\mathcal{E}^{ik})} + G_i\|\delta_k^\dagger(\mathbf{w}^{(i)}-\mathbf{w}^{(k)})\|^2_{L_2(\mathcal{E}^{ik})}.$$

We analyze the first term in detail; the second one can be bounded completely analogously. We assume that $\mathcal{E}^{ik}\subset\partial\mathcal{F}^{ij}$, where $\mathcal{F}^{ij}$ is a face common to Ω_i and Ω_j. Using formula (18), we obtain

$$\begin{aligned}
& G_i\|\delta_j^\dagger(\mathbf{w}^{(i)}-\mathbf{w}^{(j)})\|^2_{L_2(\mathcal{E}^{ik})} \leq \min(G_i,G_j)\|\mathbf{w}^{(i)}-\mathbf{w}^{(j)}\|^2_{L_2(\mathcal{E}^{ik})} \\
= \;& \min(G_i,G_j)\,\|(\mathbf{w}^{(i)}-\sum_{l=1}^{6} f_l^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)})\mathbf{r}_l)-(\mathbf{w}^{(j)}-\sum_{l=1}^{6} f_l^{\mathcal{F}^{ij}}(\mathbf{w}^{(j)})\mathbf{r}_l)\|^2_{L_2(\mathcal{E}^{ik})} \\
\leq \;& 2G_i\|\mathbf{w}^{(i)}-\sum_{l=1}^{6} f_l^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)})\mathbf{r}_l\|^2_{L_2(\mathcal{E}^{ik})} + 2G_j\|\mathbf{w}^{(j)}-\sum_{l=1}^{6} f_l^{\mathcal{F}^{ij}}(\mathbf{w}^{(j)})\mathbf{r}_l\|^2_{L_2(\mathcal{E}^{ik})}.
\end{aligned}$$

Let $\mathbf{r}^{(i)}\in\mathbf{W}^{(i)}$ again be an arbitrary rigid body mode. We can then proceed similarly as for the face contributions. For the first term on the right hand side, we obtain, using Lemma 11 and the triangular inequality,

$$\begin{aligned}
& 2G_i\|\mathbf{w}^{(i)}-\sum_{l=1}^{6} f_l^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)})\mathbf{r}_l\|^2_{L_2(\mathcal{E}^{ik})} \\
= \;& 2G_i\|(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})-\sum_{l=1}^{6} f_l^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})\mathbf{r}_l\|^2_{L_2(\mathcal{E}^{ik})} \\
\leq \;& 4G_i\|\mathbf{w}^{(i)}-\mathbf{r}^{(i)}\|^2_{L_2(\mathcal{E}^{ik})} + 4G_i\|\sum_{l=1}^{6} f_l^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})\mathbf{r}_l\|^2_{L_2(\mathcal{E}^{ik})} \\
\leq \;& C(1+\log(H_i/h_i))G_i\left(|\mathbf{w}^{(i)}-\mathbf{r}^{(i)}|^2_{H^{1/2}(\mathcal{F}^{ij})} + \frac{1}{H_i}\|\mathbf{w}^{(i)}-\mathbf{r}^{(i)}\|^2_{L_2(\mathcal{F}^{ij})}\right) \\
& \qquad\qquad + C\,G_i\sum_{l=1}^{6}|f_l^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})|^2\|\mathbf{r}_l\|^2_{L_2(\mathcal{E}^{ik})}.
\end{aligned}$$

It can be easily shown that

$$\|\mathbf{r}_m\|^2_{L_2(\mathcal{E}^{ik})} \le C\, \min(H_i, H_j) \quad m = 1, \ldots, 6, \tag{41}$$

with a positive constant C independent of h, H, and G_i. We can now proceed exactly as for the face contributions, selecting a minimizing $\mathbf{r}^{(i)}$, but now using the estimate (41) instead of (40). We obtain

$$2G_i\|\mathbf{w}^{(i)} - \sum_{l=1}^{6} f_l^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)})\mathbf{r}_l)\|^2_{L_2(\mathcal{E}^{ik})} \le C\, G_i(1 + \log(H_i/h_i))|\mathbf{w}^{(i)}|^2_{E(\mathcal{F}^{ij})}.$$

An analogous result holds for $2G_j\|\mathbf{w}^{(j)} - \sum_{l=1}^{6} f_l^{\mathcal{F}^{ij}}(\mathbf{w}^{(j)})\mathbf{r}_l)\|^2_{L_2(\mathcal{E}^{ik})}$. Thus, we finally obtain

$$\begin{aligned} G_i\|\delta_j^{\dagger}(\mathbf{w}^{(i)} - \mathbf{w}^{(j)})\|^2_{L_2(\mathcal{E}^{ik})} \quad \le \quad & C\, G_i(1 + \log(H_i/h_i))|\mathbf{w}^{(i)}|^2_{E(\mathcal{F}^{ij})} \\ & + C\, G_j(1 + \log(H_j/h_j))|\mathbf{w}^{(j)}|^2_{E(\mathcal{F}^{ij})}. \end{aligned}$$

□

8.2. Second case. We again assume that all vertices are primal, all faces are fully primal, and that each edge which is common to no more than three subdomains is treated as in Subsection 8.1. Additionally, we assume that any edge $\mathcal{E}^{ik}$, which is common to more than three subdomains is fully primal; cf. Definition 2. Thus, for such an edge, we have five linear functionals $f_m(\cdot)$ which satisfy Definition 2 and we have the property $f_m(\mathbf{w}^{(i)}) = f_m(\mathbf{w}^{(j)})\ \ \forall \mathbf{w}^{(i)} \in \mathbf{W}^{(i)}, \mathbf{w}^{(j)} \in \mathbf{W}^{(j)}$. Here, Ω_i and Ω_j form an arbitrary pair of subdomains which has the edge $\mathcal{E}^{ik}$ in common. The functionals $f_m(\cdot), m = 1, \ldots, 5$, are defined in (25). For this case, as in Subsection 8.1, we are able to prove Condition 1 with $TOL = 1$.

Lemma 18. *For all* $\mathbf{w} \in \widetilde{\mathbf{W}}$*, we have*

$$|P_\Delta \mathbf{w}|^2_{S_\varepsilon} \le C\,(1 + \log(H/h))^2 |\mathbf{w}|^2_{S_\varepsilon}.$$

Proof. We consider an arbitrary $\mathbf{w} \in \widetilde{\mathbf{W}}$. As in the proof of Lemma 17, using again formula (14), we see that it is sufficient to show

$$|P_\Delta \mathbf{w}|^2_{S} \le C\,(1 + \log(H/h))^2 |\mathbf{w}|^2_{S_\varepsilon}.$$

With $\mathbf{v}^{(i)} := (P_\Delta \mathbf{w})^{(i)}$, we again have

$$|P_\Delta \mathbf{w}|^2_S = \sum_{i=1}^{N} |\mathbf{v}^{(i)}|^2_{S^{(i)}}.$$

We cut each function $\mathbf{v}^{(i)}$ using the partition-of-unity functions $\theta_{\mathcal{F}^{ij}}, \theta_{\mathcal{E}^{ik}}$, and $\theta_{\mathcal{V}^{il}}$ and write it as a sum of terms which vanish at all the interface nodes outside individual faces, edges, and vertices, respectively; cf. (36). We note that the vertex terms vanish since all vertices are again primal. The face contribution can be analyzed as in the proof of Lemma 17 and there remains to estimate the edge contributions.

Edge terms

Here, it is sufficient to consider those edges which cannot be reduced to face estimates since those cases have already been treated in Subsection 8.1. As in the proof of Lemma 17, we have to estimate L_2-terms. By using Lemma 10, we can estimate the contribution of the edges of Ω_i to the energy of $\mathbf{v}^{(i)}$ in terms of L_2-norms over the edges. These L_2-norms are then estimated by using Lemma 11. If four subdomains, e.g., $\Omega_i, \Omega_j, \Omega_k$, and Ω_l have an edge $\mathcal{E}^{ik}$ in common, cf. Figure 4, then, according to

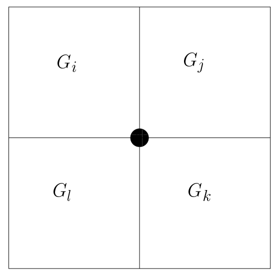

FIG. 4. *Planar cut of four subdomains meeting at an edge.*

(34), there are three contributions to the estimate of the contribution of Ω_i to $|P_D \mathbf{w}|_S$, namely,

$$
\begin{aligned}
G_i \|\delta_j^\dagger(\mathbf{w}^{(i)} - \mathbf{w}^{(j)})\|^2_{L_2(\mathcal{E}^{ik})} \quad &+ \quad G_i \|\delta_k^\dagger(\mathbf{w}^{(i)} - \mathbf{w}^{(k)})\|^2_{L_2(\mathcal{E}^{ik})} \\
&+ \quad G_i \|\delta_l^\dagger(\mathbf{w}^{(i)} - \mathbf{w}^{(l)})\|^2_{L_2(\mathcal{E}^{ik})}.
\end{aligned} \tag{42}
$$

We first analyze the second term in detail assuming that the functionals $f_m^{\mathcal{E}^{ik}}(\mathbf{w}^{(i)}) := f_m(\mathbf{w}^{(i)})$ and $f_m^{\mathcal{E}^{ik}}(\mathbf{w}^{(k)}) := f_m(\mathbf{w}^{(k)})$ have the same value on $\mathcal{E}^{ik}$ for $m = 1, \ldots, 5$. We can now proceed almost exactly as in the estimates of the edge contributions in the proof of Lemma 17. The only difference is that one rotational rigid body mode is linear dependent on the others on the edge $\mathcal{E}^{ik}$ and without restriction of generality, we assume that it is $\mathbf{r}_6$ and that it vanishes; cf. discussion before formula (25). As before for the face contributions, cf. (38), we have, on the edge $\mathcal{E}^{ik}$ and for an arbitrary rigid body mode $\mathbf{r}$, the following representation formula,

$$
\mathbf{r} = \sum_{m=1}^{5} f_m^{\mathcal{E}^{ik}}(\mathbf{r})\mathbf{r}_m. \tag{43}
$$

We note that we are free to add different multiples of $\mathbf{r}_6$ in the two subdomains since their difference will vanish on $\mathcal{E}^{ik}$. On the edge $\mathcal{E}^{ik}$, we now have

$$
\mathbf{w}^{(i)} - \mathbf{w}^{(k)} = \left(\mathbf{w}^{(i)} - \sum_{m=1}^{5} f_m^{\mathcal{E}^{ik}}(\mathbf{w}^{(i)})\mathbf{r}_m\right) - \left(\mathbf{w}^{(k)} - \sum_{m=1}^{5} f_m^{\mathcal{E}^{ik}}(\mathbf{w}^{(k)})\mathbf{r}_m\right). \tag{44}
$$

Using (43), we obtain for the first term on the right hand side and an arbitrary rigid body mode $\mathbf{r}^{(i)} \in \mathbf{W}^{(i)}$,

$$
\mathbf{w}^{(i)} - \sum_{m=1}^{5} f_m^{\mathcal{E}^{ik}}(\mathbf{w}^{(i)})\mathbf{r}_m = (\mathbf{w}^{(i)} - \mathbf{r}^{(i)}) - \sum_{m=1}^{5} f_m^{\mathcal{E}^{ik}}(\mathbf{w}^{(i)} - \mathbf{r}^{(i)})\mathbf{r}_m. \tag{45}
$$

Using formula (18), (45), and the triangular inequality, we obtain,

$$\begin{aligned}
& G_i\|\delta_k^\dagger(\mathbf{w}^{(i)}-\mathbf{w}^{(k)})\|^2_{L_2(\mathcal{E}^{ik})} \le \min(G_i,G_k)\|\mathbf{w}^{(i)}-\mathbf{w}^{(k)}\|^2_{L_2(\mathcal{E}^{ik})} \\
= \;& \min(G_i,G_k)\|(\mathbf{w}^{(i)}-\sum_{m=1}^{5} f_m^{\mathcal{E}^{ik}}(\mathbf{w}^{(i)})\mathbf{r}_m)-(\mathbf{w}^{(k)}-\sum_{m=1}^{5} f_m^{\mathcal{E}^{ik}}(\mathbf{w}^{(k)})\mathbf{r}_m)\|^2_{L_2(\mathcal{E}^{ik})} \\
\le \;& 2G_i\|\mathbf{w}^{(i)}-\sum_{m=1}^{5} f_m^{\mathcal{E}^{ik}}(\mathbf{w}^{(i)})\mathbf{r}_m\|^2_{L_2(\mathcal{E}^{ik})}+2G_k\|\mathbf{w}^{(k)}-\sum_{m=1}^{5} f_m^{\mathcal{E}^{ik}}(\mathbf{w}^{(k)})\mathbf{r}_m\|^2_{L_2(\mathcal{E}^{ik})} \\
= \;& 2G_i\|(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})-\sum_{m=1}^{5} f_m^{\mathcal{E}^{ik}}(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})\mathbf{r}_m\|^2_{L_2(\mathcal{E}^{ik})} \\
& \qquad\qquad +2G_k\|(\mathbf{w}^{(k)}-\mathbf{r}^{(i)})-\sum_{m=1}^{5} f_m^{\mathcal{E}^{ik}}(\mathbf{w}^{(k)}-\mathbf{r}^{(i)})\mathbf{r}_m\|^2_{L_2(\mathcal{E}^{ik})}.
\end{aligned}$$

We can now choose the minimizing $\mathbf{r}^{(i)}$ and $\mathbf{r}^{(k)}$ and we also note that the last two terms can be treated as the edge contributions in Section 8.1 and we obtain

$$\begin{aligned}
G_i\|\delta_k^\dagger(\mathbf{w}^{(i)}-\mathbf{w}^{(k)})\|^2_{L_2(\mathcal{E}^{ik})} \;\le\; & C\,G_i(1+\log(H_i/h_i))|\mathbf{w}^{(i)}|^2_{E(\mathcal{F}^{ij})} \\
& +C\,G_k(1+\log(H_k/h_k))|\mathbf{w}^{(k)}|^2_{E(\mathcal{F}^{jk})}.
\end{aligned}$$

The remaining two edge contributions are simpler and can be reduced to the case of face contributions as in the proof of Lemma 17.

□

8.3. Third case. In this section, we show that it is often possible to use a smaller number of (fully) primal edges and to have relatively few primal vertices. We will analyze all coefficient distributions which cannot be treated as in Subsections 8.1 and 8.2. We make the assumption that for each pair of subdomains that share a face or an edge, we have an acceptable face path which only goes through fully primal faces; cf. Definition 3. For those subdomains which share only a vertex, which is not primal, we assume that there exists an acceptable vertex path passing through fully primal faces; cf. Definition 4. We then have

LEMMA 19. *For all* $\mathbf{w}\in\widetilde{\mathbf{W}}$, *we have,*

$$|P_\Delta \mathbf{w}|^2_{S_\varepsilon} \le C\,\max(1,TOL)\,(1+\log(H/h))^2|\mathbf{w}|^2_{S_\varepsilon}.$$

Proof. We consider an arbitrary $\mathbf{w}\in\widetilde{\mathbf{W}}$. As in the proof of Lemma 17, using again formula (14), we see that it is sufficient to show that

$$|P_\Delta \mathbf{w}|^2_{S} \le C\,\max(1,TOL)\,(1+\log(H/h))^2|\mathbf{w}|^2_{S_\varepsilon}.$$

With $\mathbf{v}^{(i)} := (P_\Delta\mathbf{w})^{(i)}$, we again have

$$|P_\Delta \mathbf{w}|^2_{S} = \sum_{i=1}^{N} |\mathbf{v}^{(i)}|^2_{S^{(i)}}.$$

We cut each function $\mathbf{v}^{(i)}$ using the partition-of-unity functions $\theta_{\mathcal{F}^{ij}},\theta_{\mathcal{E}^{ik}}$, and $\theta_{\mathcal{V}^{il}}$ and write it as a sum of terms which vanish at all the interface nodes outside individual faces, edges, and vertices, respectively,

$$\mathbf{v}^{(i)} = \sum_{\mathcal{F}^{ij}\subset\partial\Omega_i} I^h(\theta_{\mathcal{F}^{ij}}\mathbf{v}^{(i)}) + \sum_{\mathcal{E}^{ik}\subset\partial\Omega_i} I^h(\theta_{\mathcal{E}^{ik}}\mathbf{v}^{(i)}) + \sum_{\mathcal{V}^{il}\in\partial\Omega_i} \theta_{\mathcal{V}^{il}}\mathbf{v}^{(i)}(\mathcal{V}^{il}) \tag{46}$$

Face terms

As before, we find from (34), that the face $\mathcal{F}^{ij}$ contributes

$$I^h(\theta_{\mathcal{F}^{ij}}\delta_j^\dagger(\mathbf{w}^{(i)}-\mathbf{w}^{(j)}))$$

and we have to estimate its $H_{00}^{1/2}(\mathcal{F}^{ij})-$norm. If the face $\mathcal{F}^{ij}$ is fully primal, we can proceed as in Section 8.1. Let us now assume that $\mathcal{F}^{ij}$ is not fully primal and that we have an acceptable face path from Ω_i to Ω_j; cf. Definition 3. For simplicity, we assume that the path passes through $\Omega_{k_1}, \Omega_{k_2}$ or more precisely through the fully primal faces $\mathcal{F}^{ik_1}, \mathcal{F}^{k_1k_2}$, and $\mathcal{F}^{k_2j}$; the path could of course also lead through more subdomains and fully primal faces, respectively. For $\mathcal{F}^{ij}$ and each of the fully primal faces on the acceptable face path, we introduce a basis of shifted rigid body modes such that the origin is shifted to or close to the center of the face; cf. (4). We denote these bases by

$$(\mathbf{r}_l^{\mathcal{F}^{ij}})_{l=1,\ldots,6}, \quad (\mathbf{r}_l^{\mathcal{F}^{ik_1}})_{l=1,\ldots,6}, \quad (\mathbf{r}_l^{\mathcal{F}^{k_1k_2}})_{l=1,\ldots,6}, \quad (\mathbf{r}_l^{\mathcal{F}^{k_2j}})_{l=1,\ldots,6}.$$

These bases are also used in the construction of the functionals $f_m^{\mathcal{F}^{ij}}, f_m^{\mathcal{F}^{ik_1}}, f_m^{\mathcal{F}^{k_1k_2}}$, and $f_m^{\mathcal{F}^{k_2j}}$; cf. Definition 3 and the discussion that follows. We obtain

$$\begin{aligned}
&\mathbf{w}^{(i)}-\mathbf{w}^{(j)} \\
= &\left(\mathbf{w}^{(i)}-\sum_{m=1}^{6} f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)})\mathbf{r}_m^{\mathcal{F}^{ij}}\right)+\left(\sum_{m=1}^{6} f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)})\mathbf{r}_m^{\mathcal{F}^{ij}}-\sum_{m=1}^{6} f_m^{\mathcal{F}^{ik_1}}(\mathbf{w}^{(i)})\mathbf{r}_m^{\mathcal{F}^{ik_1}}\right) \\
&+\left(\sum_{m=1}^{6} f_m^{\mathcal{F}^{ik_1}}(\mathbf{w}^{(k_1)})\mathbf{r}_m^{\mathcal{F}^{ik_1}}-\sum_{m=1}^{6} f_m^{\mathcal{F}^{k_1k_2}}(\mathbf{w}^{(k_1)})\mathbf{r}_m^{\mathcal{F}^{k_1k_2}}\right) \\
&+\left(\sum_{m=1}^{6} f_m^{\mathcal{F}^{k_1k_2}}(\mathbf{w}^{(k_2)})\mathbf{r}_m^{\mathcal{F}^{k_1k_2}}-\sum_{m=1}^{6} f_m^{\mathcal{F}^{k_2j}}(\mathbf{w}^{(k_2)})\mathbf{r}_m^{\mathcal{F}^{k_2j}}\right) \\
&+\left(\sum_{m=1}^{6} f_m^{\mathcal{F}^{k_2j}}(\mathbf{w}^{(j)})\mathbf{r}_m^{\mathcal{F}^{k_2j}}-\sum_{m=1}^{6} f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(j)})\mathbf{r}_m^{\mathcal{F}^{ij}}\right)+\left(\sum_{m=1}^{6} f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(j)})\mathbf{r}_m^{\mathcal{F}^{ij}}-\mathbf{w}^{(j)}\right).
\end{aligned}$$

The first and the last term on the right hand side can be estimated as the face contributions in Section 8.1. Let us now consider

$$\sum_{m=1}^{6} f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)})\mathbf{r}_m^{\mathcal{F}^{ij}}-\sum_{m=1}^{6} f_m^{\mathcal{F}^{ik_1}}(\mathbf{w}^{(i)})\mathbf{r}_m^{\mathcal{F}^{ik_1}} \tag{47}$$

in detail; the other intermediate sums can be estimated analogously. Let $\mathbf{r}^{(i)} \in \mathbf{W}^{(i)}$ be an arbitrary rigid body mode. Then, we have the following representations with respect to the two different bases of rigid body modes related to the two different faces $\mathcal{F}^{ij}$ and $\mathcal{F}^{ik_1}$,

$$\mathbf{r}^{(i)}=\sum_{l=1}^{6} f_l^{\mathcal{F}^{ij}}(\mathbf{r}^{(i)})\mathbf{r}_l^{\mathcal{F}^{ij}}, \quad \mathbf{r}^{(i)}=\sum_{l=1}^{6} f_l^{\mathcal{F}^{ik_1}}(\mathbf{r}^{(i)})\mathbf{r}_l^{\mathcal{F}^{ik_1}}.$$

Using these two different representations, we obtain from (47) by subtracting and adding $\mathbf{r}^{(i)}$,

$$\sum_{m=1}^{6} f_m^{\mathcal{F}^{ij}}(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})\mathbf{r}_m^{\mathcal{F}^{ij}}-\sum_{m=1}^{6} f_m^{\mathcal{F}^{ik_1}}(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})\mathbf{r}_m^{\mathcal{F}^{ik_1}}.$$

The $H_{00}^{1/2}(\mathcal{F}^{ij})$-norm of the finite element interpolation of the first sum multiplied by $\theta_{\mathcal{F}^{ij}}$ can be estimated as in Section 8.1. For the second sum, the only difference is that we use

$$\|I^h(\theta_{\mathcal{F}^{ij}}\mathbf{r}_m^{\mathcal{F}^{ik_1}})\|^2_{H_{00}^{1/2}(\mathcal{F}^{ij})} \leq C\,H_i\,(1+\log(H_i/h_i)),$$

which follows from Lemma 8; cf. also (40). Using a separate rigid body mode to shift in each subdomain on the path, we can proceed completely analogously as in Section 8.1 and obtain

$$\begin{aligned}\|I^h(\theta_{\mathcal{F}^{ij}}\delta_j^\dagger(\mathbf{w}^{(i)}-\mathbf{w}^{(j)}))\|^2_{H_{00}^{1/2}(\mathcal{F}^{ij})} \quad \leq \quad & C\,(1+\log(H/h))^2\left(|\mathbf{w}^{(i)}|^2_{E(\partial\Omega_i)}+|\mathbf{w}^{(j)}|^2_{E(\partial\Omega_j)}\right.\\ & \left.+TOL*\left(|\mathbf{w}^{(k_1)}|^2_{E(\partial\Omega_{k_1})}+|\mathbf{w}^{(k_2)}|^2_{E(\partial\Omega_{k_2})}\right)\right).\end{aligned}$$

Edge terms

Let us now consider an edge $\mathcal{E}^{ik}$ which is not fully primal, cf. Definition 2. Here, we again need the concept of an acceptable face path, cf. Definition 3, and we assume that we have such a path through the fully primal faces $\mathcal{F}^{ij_1}$, $\mathcal{F}^{j_1j_2}$, $\mathcal{F}^{j_2j_3}$, and $\mathcal{F}^{j_3k}$. The basis of rigid body modes $(\mathbf{r}_l^{\mathcal{E}^{ik}})_{l=1,\ldots,6}$ has only five linearly independent elements when restricted to the edge $\mathcal{E}^{ik}$, since one rotation is linearly dependent on the others or even vanishes. Using an appropriate change of coordinates such that $\mathcal{E}^{ik}$ coincides with the x_1-axis, we can assume that the rotation $\mathbf{r}_6^{\mathcal{E}^{ik}}$ vanishes.

For the faces, we obviously have six functionals and we assume that they are all built using the basis $\mathbf{r}_l^{\mathcal{E}^{ik}}, l=1,\ldots,6$. Since $\mathbf{r}_6^{\mathcal{E}^{ik}}$ vanishes on $\mathcal{E}^{ik}$, we have, for an arbitrary rigid body mode $\mathbf{r}^{(i)}\in\mathbf{W}^{(i)}$,

$$\mathbf{r}^{(i)} = \sum_{m=1}^{5} f_m^{\mathcal{F}^{ij_1}}(\mathbf{r}^{(i)})\mathbf{r}_m^{\mathcal{E}^{ik}} \tag{48}$$

on $\mathcal{E}^{ik}$. Similarly, for an arbitrary rigid body mode $\mathbf{r}^{(j_1)}\in\mathbf{W}^{(j_1)}$, we have the expansion on $\mathcal{E}^{ik}$

$$\mathbf{r}^{(j_1)} = \sum_{m=1}^{5} f_m^{\mathcal{F}^{j_1j_2}}(\mathbf{r}^{(j_1)})\mathbf{r}_m^{\mathcal{E}^{ik}}. \tag{49}$$

Furthermore, we have for $p=1,2,3$,

$$\mathrm{TOL}*G_{j_p}\geq\min(G_i,G_k). \tag{50}$$

The edge contributions are again given as in (42). Considering, as before, the second edge term, associated with two subdomains sharing an edge but not a face, we obtain

$$G_i\|\delta_k^\dagger(\mathbf{w}^{(i)}-\mathbf{w}^{(k)})\|^2_{L_2(\mathcal{E}^{ik})}\leq\min(G_i,G_k)\,\|\mathbf{w}^{(i)}-\mathbf{w}^{(k)}\|^2_{L_2(\mathcal{E}^{ik})}$$

We have

$$
\begin{aligned}
\mathbf{w}^{(i)} - \mathbf{w}^{(k)} &= \left(\mathbf{w}^{(i)} - \sum_{m=1}^{5} f_m^{\mathcal{F}^{ij_1}}(\mathbf{w}^{(i)})\mathbf{r}_m^{\mathcal{E}^{ik}}\right) \\
&+ \left(\sum_{m=1}^{5} f_m^{\mathcal{F}^{ij_1}}(\mathbf{w}^{(j_1)})\mathbf{r}_m^{\mathcal{E}^{ik}} - \sum_{m=1}^{5} f_m^{\mathcal{F}^{j_1 j_2}}(\mathbf{w}^{(j_1)})\mathbf{r}_m^{\mathcal{E}^{ik}}\right) \\
&+ \left(\sum_{m=1}^{5} f_m^{\mathcal{F}^{j_1 j_2}}(\mathbf{w}^{(j_2)})\mathbf{r}_m^{\mathcal{E}^{ik}} - \sum_{m=1}^{5} f_m^{\mathcal{F}^{j_2 j_3}}(\mathbf{w}^{(j_2)})\mathbf{r}_m^{\mathcal{E}^{ik}}\right) \\
&+ \left(\sum_{m=1}^{5} f_m^{\mathcal{F}^{j_2 j_3}}(\mathbf{w}^{(j_3)})\mathbf{r}_m^{\mathcal{E}^{ik}} - \sum_{m=1}^{5} f_m^{\mathcal{F}^{j_3 k}}(\mathbf{w}^{(j_3)})\mathbf{r}_m^{\mathcal{E}^{ik}}\right) \\
&+ \left(\sum_{m=1}^{5} f_m^{\mathcal{F}^{j_3 k}}(\mathbf{w}^{(k)})\mathbf{r}_m^{\mathcal{E}^{ik}} - \mathbf{w}^{(k)}\right).
\end{aligned} \tag{51}
$$

Using (48), we obtain on $\mathcal{E}^{ik}$

$$
\mathbf{w}^{(i)} - \sum_{m=1}^{5} f_m^{\mathcal{F}^{ij_1}}(\mathbf{w}^{(i)})\mathbf{r}_m^{\mathcal{E}^{ik}} = (\mathbf{w}^{(i)} - \mathbf{r}^{(i)}) - \sum_{m=1}^{5} f_m^{\mathcal{F}^{ij_1}}(\mathbf{w}^{(i)} - \mathbf{r}^{(i)})\mathbf{r}_m^{\mathcal{E}^{ik}}.
$$

Hence, the $L_2(\mathcal{E}^{ik})$-norm of the first term on the right hand side of (51) can be estimated as in Section 8.1 using a minimizing rigid body mode $\mathbf{r}^{(i)}$. We can proceed analogously for the last term on the right hand side of (51). Let us now consider the third term; the remaining terms can be treated analogously. For an arbitrary rigid body mode $\mathbf{r}^{(j_1)} \in \mathbf{W}^{(j_1)}$, using (49), we obtain on $\mathcal{E}^{ik}$

$$
\begin{aligned}
&\sum_{m=1}^{5} f_m^{\mathcal{F}^{ij_1}}(\mathbf{w}^{(j_1)})\mathbf{r}_m^{\mathcal{E}^{ij_1}} - \sum_{m=1}^{5} f_m^{\mathcal{F}^{j_1 j_2}}(\mathbf{w}^{(j_1)})\mathbf{r}_m^{\mathcal{E}^{ik}} \\
= \; &(\sum_{m=1}^{5} f_m^{\mathcal{F}^{ij_1}}(\mathbf{w}^{(j_1)})\mathbf{r}_m^{\mathcal{E}^{ij_1}} - \mathbf{r}^{(j_1)}) - (\sum_{m=1}^{5} f_m^{\mathcal{F}^{j_1 j_2}}(\mathbf{w}^{(j_1)})\mathbf{r}_m^{\mathcal{E}^{ik}} - \mathbf{r}^{(j_1)}) \\
= \; &\sum_{m=1}^{5} f_m^{\mathcal{F}^{ij_1}}(\mathbf{w}^{(j_1)} - \mathbf{r}^{(j_1)})\mathbf{r}_m^{\mathcal{E}^{ij_1}} - \sum_{m=1}^{5} f_m^{\mathcal{F}^{j_1 j_2}}(\mathbf{w}^{(j_1)} - \mathbf{r}^{(j_1)})\mathbf{r}_m^{\mathcal{E}^{ik}}.
\end{aligned}
$$

The $L_2(\mathcal{E}^{ik})$-norm of the resulting two terms can now be estimated as in Sections 8.1 and 8.2 using a minimizing rigid body mode $\mathbf{r}^{(j_1)}$ and we obtain, using (50),

$$
\begin{aligned}
G_i \|\delta_k^\dagger(\mathbf{w}^{(i)} - \mathbf{w}^{(k)})\|^2_{L_2(\mathcal{E}^{ik})} &\leq C\,(1 + \log(H_i/h_i))\, G_i\, |\mathbf{w}^{(i)}|^2_{E(\partial\Omega_i)} \\
&+ C\,(1 + \log(H_{j_1}/h_{j_1}))\, G_{j_1} * TOL\, |\mathbf{w}^{(j_1)}|^2_{E(\partial\Omega_{j_1})} \\
&+ C\,(1 + \log(H_{j_2}/h_{j_2}))\, G_{j_2} * TOL\, |\mathbf{w}^{(j_2)}|^2_{E(\partial\Omega_{j_2})} \\
&+ C\,(1 + \log(H_{j_3}/h_{j_3}))\, G_{j_3} * TOL\, |\mathbf{w}^{(j_3)}|^2_{E(\partial\Omega_{j_3})} \\
&+ C\,(1 + \log(H_k/h_k))\, G_k\, |\mathbf{w}^{(k)}|^2_{E(\partial\Omega_k)}.
\end{aligned}
$$

Vertex terms

Finally, we estimate the terms resulting from the vertices. We have, according to (34),

$$\begin{aligned} G_i|\mathbf{v}^{(i)}(\mathcal{V}^{i\ell})\theta_{\mathcal{V}^{i\ell}}|^2_{H^{1/2}(\partial\Omega_i)} &\leq C \sum_{j\in\mathcal{N}_{\mathcal{V}^{i\ell}}} G_i(\delta_j^\dagger)^2|(\mathbf{w}^{(i)}(\mathcal{V}^{i\ell})-\mathbf{w}^{(j)}(\mathcal{V}^{i\ell}))\theta_{\mathcal{V}^{i\ell}}|^2_{H^{1/2}(\partial\Omega_i)} \\ &\leq C \sum_{j\in\mathcal{N}_{\mathcal{V}^{i\ell}}} \min(G_i,G_j)|(\mathbf{w}^{(i)}(\mathcal{V}^{i\ell})-\mathbf{w}^{(j)}(\mathcal{V}^{i\ell}))\theta_{\mathcal{V}^{i\ell}}|^2_{H^{1/2}(\partial\Omega_i)}. \end{aligned}$$

We proceed by considering each pair of substructures separately. Let Ω_i, Ω_l be such a pair and assume that we have an acceptable vertex path through fully primal faces $\mathcal{F}^{ij_1}, \mathcal{F}^{j_1j_2}$, and $\mathcal{F}^{j_2l}$; cf. Definition 4. We have

$$\begin{aligned} \mathbf{w}^{(i)}-\mathbf{w}^{(l)} &= \mathbf{w}^{(i)} - \sum_{m=1}^{3} f_m^{\mathcal{F}^{ij_1}}(\mathbf{w}^{(i)})\mathbf{r}_m^{\mathcal{V}^{il}} \\ &\quad + \sum_{m=1}^{3} f_m^{\mathcal{F}^{ij_1}}(\mathbf{w}^{(j_1)})\mathbf{r}_m^{\mathcal{V}^{il}} - \sum_{m=1}^{3} f_m^{\mathcal{F}^{j_1j_2}}(\mathbf{w}^{(j_1)})\mathbf{r}_m^{\mathcal{V}^{il}} \\ &\quad + \sum_{m=1}^{3} f_m^{\mathcal{F}^{j_1j_2}}(\mathbf{w}^{(j_2)})\mathbf{r}_m^{\mathcal{V}^{il}} - \sum_{m=1}^{3} f_m^{\mathcal{F}^{j_2l}}(\mathbf{w}^{(j_2)})\mathbf{r}_m^{\mathcal{V}^{il}} \\ &\quad + \sum_{m=1}^{3} f_m^{\mathcal{F}^{j_2l}}(\mathbf{w}^{(l)})\mathbf{r}_m^{\mathcal{V}^{il}} - \mathbf{w}^{(l)}. \end{aligned} \tag{52}$$

Let $\mathbf{r}^{(i)} \in \mathbf{W}^{(i)}$ be an arbitrary rigid body mode. We then have

$$\mathbf{r}^{(i)}(\mathcal{V}^{il}) = \sum_{m=1}^{3} f_m^{\mathcal{F}^{ij_1}}(\mathbf{r}^{(i)})\mathbf{r}_m^{\mathcal{V}^{ij_1}}(\mathcal{V}^{il}).$$

Let us now consider

$$\begin{aligned} &\mathbf{w}^{(i)}(\mathcal{V}^{il}) - \sum_{m=1}^{3} f_m^{\mathcal{F}^{ij_1}}(\mathbf{w}^{(i)})\mathbf{r}_m^{\mathcal{V}^{il}}(\mathcal{V}^{il}) \\ = \;& (\mathbf{w}^{(i)}-\mathbf{r}^{(i)})(\mathcal{V}^{il}) - \sum_{m=1}^{3} f_m^{\mathcal{F}^{ij_1}}(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})\mathbf{r}_m^{\mathcal{V}^{ij_1}}(\mathcal{V}^{il}). \end{aligned} \tag{53}$$

Applying Lemma 12, we obtain

$$|(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})(\mathcal{V}^{il})\theta_{\mathcal{V}^{il}}|^2_{H^{1/2}(\partial\Omega_i)} \leq C\,(|\mathbf{w}^{(i)}-\mathbf{r}^{(i)}|^2_{H^{1/2}(\partial\Omega_i)} + 1/H_i\|\mathbf{w}^{(i)}-\mathbf{r}^{(i)}\|^2_{L_2(\partial\Omega_i)})$$

Taking the infimum over all rigid body modes and applying Lemma 7, we obtain

$$|(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})(\mathcal{V}^{il})\theta_{\mathcal{V}^{il}}|^2_{H^{1/2}(\partial\Omega_i)} \leq C\,|\mathbf{w}^{(i)}|^2_{E(\partial\Omega_i)}.$$

For the second term on the right hand side of (53), we obtain, for a minimizing $\mathbf{r}^{(i)}$,

$$\begin{aligned} & |f_m^{\mathcal{F}^{ij_1}}(\mathbf{w}^{(i)}-\mathbf{r}^{(i)})\mathbf{r}_m^{\mathcal{V}^{ij_1}}(\mathcal{V}^{il})\theta_{\mathcal{V}^{il}}|^2_{H^{1/2}(\partial\Omega_i)} \\ \leq\;& C\,h_i/H_i\,(1+\log(H_i/h_i))\,(|\mathbf{w}^{(i)}-\mathbf{r}^{(i)}|^2_{H^{1/2}(\partial\Omega_i)} + 1/H_i\|\mathbf{w}^{(i)}-\mathbf{r}^{(i)}\|^2_{L_2(\partial\Omega_i)}) \\ \leq\;& C\,h_i/H_i\,(1+\log(H_i/h_i))\,|\mathbf{w}^{(i)}|^2_{E(\partial\Omega_i)}. \end{aligned}$$

Proceeding analogously for the remaining terms on the right hand side of (52) and using

$$TOL * G_j \geq \frac{h_j}{H_j} \min(G_i, G_l), \tag{54}$$

we obtain

$$\begin{aligned} & \min(G_i, G_l)|(\mathbf{w}^{(i)}(\mathcal{V}^{il}) - \mathbf{w}^{(l)}(\mathcal{V}^{il}))\theta_{\mathcal{V}^{il}}|^2_{H^{1/2}(\partial\Omega_i)} \\ \leq \quad & C\,(1 + \log(H_i/h_i))\, G_i\, |\mathbf{w}^{(i)}|^2_{E(\partial\Omega_i)} + C\,(1 + \log(H_{j_1}/h_{j_1}))\, G_{j_1} * TOL\, |\mathbf{w}^{(j_1)}|^2_{E(\partial\Omega_{j_1})} \\ + \quad & C\,(1 + \log(H_{j_2}/h_{j_2}))\, G_{j_2} * TOL\, |\mathbf{w}^{(j_2)}|^2_{E(\partial\Omega_{j_2})} + C\,(1 + \log(H_l/h_l))\, G_l\, |\mathbf{w}^{(l)}|^2_{E(\partial\Omega_k)}. \end{aligned}$$

Let us note that more general acceptable vertex paths can be analyzed analogously.

$\square$

8.4. Algorithmic selection of primal constraints. The concepts of an acceptable path might appear to be fairly complicated. We will therefore explore the possibility of developing a set of relatively simple rules which would guarantee that our asssumptions related to acceptable paths will be satisfied. We recall that the simplest way to assure that a face has an acceptable face path, is to introduce six linearly independent edge constraints across the face. Similarly, we can meet the requirement for an edge by making it fully primal. However, our goal is to be selective and to try to identify a small and effective primal constraint set. We will also explore, in greater detail, the effects of possible long paths; we will be able to exploit what we have learned from our proofs in the previous subsection.

We first note that the selection of a linearly independent set of constraints for a fully primal face can be automated relatively simply. In the case of a quadrilateral face and using only averages over the displacement components, there are twelve functionals to choose from. We can then construct a six by twelve matrix of values obtained by evaluating all these functionals at the basis elements of the space of rigid body modes. A QR factorization with column pivoting, selecting the remaining column vector of largest norm in each step, should quickly help us select an appropriate set of six constraints. If we previously have introduced some constraints, we should order them first and use column pivoting only for the remaining edge averages; we can stop when we have found six functionals which are robustly linearly independent. We can also use the full set of five constraints that might already have been introduced for a fully primal edge as a point of departure.

Concerning subdomain edges, we note that mesh partitioners, in our [11] and other people's experience [17], often result in having a vast majority of the edges which are common to only three subdomains; cf. Subsection 8.1. There we proved that the edge terms could be reduced to face estimates for the case depicted in Figure 3 if the three faces are all fully primal. We can now show that the same is true under the weaker assumption that the three faces each have an acceptable face path. The proof is straightforward and can be modelled on the proofs of Subsection 8.3.

In case there are relatively few edges common to more than three subdomains, it would be simple and reasonable to make them all fully primal by introducing sets of five edge constraints. Alternatively, we could inspect the coefficients of the adjacent subdomains. If an acceptable face path cannot be found involving only these subdomains, we would make the edge fully primal; if such a path through fully primal faces is found, this would not be necessary.

We will now outline a strategy of selecting fully primal faces. We will introduce an undirected graph where the nodes represent subdomains and two subdomains that share a fully primal face are connected by an edge. We begin the construction of the graph, and the selection of fully primal faces, by sorting the subdomains according to decreasing values of the Young's moduli; we will activate them one by one in this order. At any time of the process, we will have one or several connected components of the graph built from activated subdomains where two subdomains will belong to the same component if there is a path from one to the other through subdomains and fully primal faces. When a subdomain is activated, it can have one or several faces in common with previously activated subdomains or it will create a new component of the graph. In the latter case, we do not introduce any new fully primal faces. If the new subdomain has a face or several faces in common with just one component, we make one of these faces fully primal by adding suitable edge average constraints to the face common to the new subdomain and the old subdomain which has the smallest index. If the current subdomain has a face or faces in common with subdomains of several components, we make one of these fully primal for each of the components, again always selecting the subdomains with the lowest index; in this case the number of components of the graph will decrease.

It is easy to see, by counting nodes and edges of the graph, that we have a forest throughout and that at the end, we have a tree. Thus, there is always a unique path from any subdomain to any other in the same component of the graph. We also note that at any stage of the activation procedure, any path from a subdomain to a next neighbor across a face which has not been selected as fully primal, will pass exclusively through subdomains with Young's modulii which are larger or equal to that of the new subdomain being considered. The existence of such a path follows from the fact that, according to the rules outlined above, there will be another, fully primal face of the same subdomain which connects it to the component of the graph of the subdomain across the first face. We also note that in this process additional faces might become fully primal when we introduce constraints on the boundary of neighboring subdomains; after adding the corresponding edges to the graph, we would no longer have a tree.

By the process just outlined, we are thus able to select a modest number of constraints, in fact fewer than six times the number of subdomains. As we already have shown, there will be no problem with the Young's moduli on the paths related to the faces which are not fully primal. But we also have to concern ourselves with the possible excessive length of such paths. But we also note that our discussion will in fact allow us to refine the arguments of Subsection 8.3. We first note that a fully primal face, shared by Ω_i and Ω_j, will contribute $\min(G_i, G_j)/G_i$ times the energy attributed to Ω_i and $\min(G_i, G_j)/G_j$ times that of Ω_j. We can systematically keep track of these terms when considering the contributions from faces and edges to the overall estimate. If the face between Ω_i and Ω_j is not fully primal and Ω_k is part of the path related to that face, we should add $\min(G_i, G_j)/G_k$ to the tally associated with Ω_k. What matters is the maximum tally over all the subdomains. If the tally is too large, e.g., if too many paths pass through a particular subdomain, then we should develop a strategy of selectively increasing the number of fully primal faces; such an action will eliminate all contributions from that face to any subdomain except the two that share the face.

We could organize this part as follows. After initializing all tallies to zero, we inspect each face of the interface one by one. If a face $\mathcal{F}^{ij}$ between Ω_i and Ω_j is fully primal, we add $\min(G_i, G_j)/G_i$ and $\min(G_i, G_j)/G_j$, respectively, to the tallies of Ω_i and Ω_j. If $\mathcal{F}^{ij}$ is not fully primal, there is at least one path from Ω_i to Ω_j and we add $\min(G_i, G_j)/G_k$ to the tally of Ω_k if Ω_k is part of the path. If this action will cause any tally to exceed a tolerance, we should instead make $\mathcal{F}^{ij}$ fully primal by adding enough constraints to make $\mathcal{F}^{ij}$ fully primal. This part of the computation should be followed or preceeded by an inspection of all edges of the interface, adding to the tallies of the subdomains in a quite similar way. Clearly, many different variants are possible.

Finally, we should inspect the vertices one by one and the Young's moduli of the subdomains that meet at the vertices. If an acceptable vertex path cannot be found using only these subdomains, we suggest that the vertex be made primal.

We hope that we soon will get practical experience with the ideas outlined in this subsection. We note that we so far have used only quite elementary graph theory; more elaborate tools and algorithms might help in the choice of a small but poweful set of primal constraints. We also note that we should reconsider the rules of selecting fully primal faces, taking the areas of the faces as well as the Young's moduli into account. Relaxing the order of the subdomains somewhat could also lead to trees with fewer levels and thus shorter paths between the subdomains.

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