Sweden

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78 Years of Ramsey $R(3, k)$

Joel Spencer

The origins of the paper go back to the early thirties. We had a very close circle of young mathematicians, foremost among them Erdős, Turán and Gallai; friendships were forged which became the most lasting that I have ever known and which outlived the upheavals of the thirties, a vicious world war and our scattering to the four corners of the world. I [. . .] often joined the mathematicians at weekend excursions in the charming hill country around Budapest and (in the summer) at open air meetings on the benches of the city park.

George Szekeres

$R(3, k)$ Lower Bounds

$R(3, k) \geq n$ if there *exists* a graph G with

- n vertices
- no triangles
- no independent set I of size k

$R(3,k)$ Upper Bounds

$R(3,k) \leq n$ if:

Given *any* G with

- n vertices

- no triangles

one can *find*

an independent set I of size k.

The Happy Ending Paper

Winter 1932-3

Paul Erdős, Esther Klein, George Szekeres

A Combinatorial Problem in Geometry

E-Sz, 1935

(Re)proof of Ramsey's Theorem

(Implicit):

$$R(l,k) \le \binom{k+l-1}{l-1}$$

So: $R(3,k) = O(k^2)$

$R(3,k) \le k^2$ Algorithmically

`IF` any $\deg(v) \ge k$ take neighbors

`ELSE` Greedy Algorithm:

Order vertices arbitrarily

Add to Ind Set if possible

(More care: $R(3,k) \le \binom{k+2}{2}$)

The "Birth" of the Probabilistic Method

April 1946. BAMS 1947

Erdős: If

$$\binom{n}{k}2^{1-\binom{k}{2}} < 1$$

then $R(k,k) > n$. That is, there *exists* an n vertex graph with neither clique nor independent set of size k

(Modern) Proof: Color randomly!

Calculation: $R(k,k) = \Omega(\sqrt{k}\sqrt{2}^k)$

An Erdős Gem

Graph Theory and Probability II

Canad J Math 13 (1961), 346-352

Theorem: $R(3,k) = \Omega(k^2 \ln^{-2} k)$

n vertex, no Δ, no $|I| = x := A\sqrt{n}\ln n$

(Modern:) $G(n,p)$, $p = \epsilon n^{-1/2}$

I *good* if it has internal edge $\{x,y\}$ not extendable to triangle $\{x,y,z\}$ with $z \notin I$.

Lemma (Hard!) $\Pr[I \text{ not good}] \ll \binom{n}{x}^{-1}$

Corollary: `whp` all I good

Erdős Magic: There *exists* G with all I good

Greedy Algorithm on G gives desired G^*

Lower Bound

Graver-Yackel (1968) $R(3,k)=\Omega(k^2\ln\ln k/(\ln k))$

Ajtai, Komlós, Szemerédi (1980)

$$R(3,k)=O(\frac{k^2}{\ln k})$$

AKSz: n vertices, no Δ, average degree $k \Rightarrow$ exists ind I

$$|I| \geq \epsilon\frac{n}{k}\ln k$$

Idea: Add "typical" v to I so density of remaining G^- goes down.

A Differential Equation

$t = \frac{n}{k}x$ chosen, $S(t)n$ remain

Average degree (!?) $kS(t)$

Choose 1, Delete $kS(t)$.

$S(t+\frac{k}{n}) - S(t) \sim -\frac{k}{n}S(t)$

$S'(t) = -S(t)$. $S(t) = e^{-t}$

Continue until (??) $t \sim \ln k$.

Final $|I| \geq \frac{n}{k}\ln k$

The Lovász Local Lemma

Events A_i, $i \in \Omega$. Dependency graph D on Ω:

A_i mutually independent of A_j with $\{j, i\} \notin D$.

LLL: If [. . .] $\wedge \overline{A_i} \neq \emptyset$

$G(n, p)$ underlying space

$|S| = 3$, A_S: S is triangle

$|T| = k$, A_T: T is independent

Calculation (JS) $R(3, k) = \Omega(k^2 \ln^{-2} k)$

The Random Greedy Algorithm

Erdős, Suen, Winkler, 1995

n vertices, no Δ.

Random Greedy to Find I

Improved constant in 1961 Result

JS (unpublished):

$$R(3,k) \gg \frac{k^2}{\ln^2 k}$$

Ramsey Resolved!

Jeong Han Kim

R S & A, vol 7 (1995) 173-207

The Ramsey Number $R(3, t)$ Has Order

of Magnitude $t^2/\log t$

Proof: Nibble + Martingales

+ Cleverness + Differential Equations

+ Diligence + . . .

Fulkerson Prize 1997

A Memorable Moment

March 6, 1998

University of Sydney

Title: 60 Years of Ramsey $R(3, k)$

Speaker: JS

First Row: Esther Klein, George Szekeres

And The Beat Goes On . . .

Bohman: Random Greedy Works!

Random Greedy Δ-free on n vertices gives `whp` G with $\Theta(n^{3/2}\sqrt{\ln n})$ edges and no independent I, $|I| = k = C\sqrt{n \ln n}$.

Time t when $tn^{3/2}$ pairs accepted.

uv IN if already in graph

uv OPEN if not in but can be added.

uv CLOSED if w with uw, vw in.

$X_{uv} =$ number w with uw, vw both open.

$Y_{uv} =$ number w with uw, vw open/in

$Q =$ number of open

Scaling:

$X_{uv} \sim x(t)n$, $Y_{uv} \sim y(t)\sqrt{n}$, $Q \sim q(t)n^2$

Differential Equations

$x'(t) \sim 2x(t)y(t)/q(t)$

$q'(t) = -y(t)$

$y'(t) = -\frac{y^2(t)+2x(t)}{q(t)}$

Solution:

$x(t) = e^{-8t^2}$, $y(t) = 4te^{-4t^2}$, $q(t) = \frac{1}{2}e^{-4t^2}$

Same as for $G(n, p)$ at that density

$x(t)$: Losing open/open pairs

$X_{uv} =$ number w with uw, vw both open.

Add edge: $t \leftarrow t + n^{-3/2}$

Pick w with uw, vw open ($x(t)n$)

Pick u or v, say v (2)

Pick z with zw open, zv in (or reverse) ($y(t)\sqrt{n}$)

Select zw. ($(q(t)n^2)^{-1}$)

Now vw closed, $X_{uv} \leftarrow X_{uv} - 1$

Expected change: $(-1)x(t)n2y(t)\sqrt{n}/(q(t)n^2)$

$$x(t + n^{-3/2}) - x(t) = -[2x(t)y(t)/q(t)]n^{-3/2}$$

$x'(t) \sim 2x(t)y(t)/q(t)$

$q(t)$: Losing open pairs

Pick open uv.

uz open, vz in (or reverse) $(y(t)\sqrt{n})$

Now uz closed

Expected change: $(-1)y(t)\sqrt{n}$

$$q(t+n^{-3/2}) - q(t) = -y(t)n^{-3/2}$$

$q'(t) = -y(t)$

$y(t)$: Gaining and Losing

uw, vw open ($x(t)n$)

Pick one ($2/(q(t)n^2)$)

$Y_{uv} \leftarrow Y_{uv} + 1$

uw in, vw open (or reverse) ($y(t)\sqrt{n}$)

zw in, zv open (or reverse) ($y(t)\sqrt{n}$)

Pick zv ($1/(q(t)n^2)$)

vw now closed, $Y_{uv} \leftarrow Y_{uv} - 1$

$$y(t + n^{-3/2}) - y(t) = [-\frac{y^2(t)}{q(t)} + \frac{2x(t)}{q(t)}]n^{-3/2}$$

$y'(t) = -\frac{y^2(t)+2x(t)}{q(t)}$

Not So Easy

Wormald Method: Random Processes via DE

Big Problem: Stability

Want DE to work until $t = \epsilon\sqrt{\ln n}$

- Martingales
- Expanding (in t) Error Bounds

Set I, $|I| = k = C\sqrt{n \ln n}$

Always has "right" number of open edges

Probability no edge ever very small

Hard part: Failure $\times \binom{n}{k} = o(1)$

It is six in the morning.

The house is asleep.

Nice music is playing.

I prove and conjecture.

– Paul Erdős, in letter to Vera Sós