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# THE

# JANSON INEQUALITIES

Joel Spencer

Any new possibility that existence acquires,
even the least likely,
transforms everything about existence.
– Milan Kundera

The Poisson Paradigm

Let $Z$ count

Many

Rare

Mostly Independent

Events with

$$E[Z] \sim \mu$$

Then

$$\Pr[Z = 0] \sim e^{-\mu}$$

Random Graph $G(n, p)$

$Z =$ number of triangles

Erdős, Rényi:

$$E[Z] \to \mu \Longrightarrow \Pr[Z = 0] \to e^{-\mu}$$

That is, $p = \frac{c}{n}$ with $c$ *constant* $\Rightarrow$

$$\Pr[Z = 0] \to e^{-c^3/6}$$

Proof: Show all moments correct.

But what if $p \gg n^{-1}$???

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The Janson Inequality

$R \subseteq \Omega$

$\Pr[r \in R] = p_r$ mutually independent

$A_\alpha \subseteq \Omega$, $\alpha \in I$

$B_\alpha$ "bad" event $A_\alpha \subseteq R$

$\mu = \sum \Pr[B_\alpha]$

$M = \prod(1 - \Pr[B_\alpha])$ (often $\sim e^{-\mu}$)

$\alpha \sim \beta$: $A_\alpha \cap A_\beta \neq \emptyset$

$$\Delta = \sum_{\alpha \sim \beta} \Pr[B_\alpha \wedge B_\beta]$$

JANSON!

$$M \leq \Pr[\wedge \overline{B_\alpha}] \leq e^{-\mu + (\Delta/2)}$$

Extended Janson: If $\Delta \geq \mu$ then

$$\Pr[\wedge \overline{B_\alpha}] \leq e^{-\mu^2/2\Delta}$$

No Triangles

$$\Delta = \sum_{\alpha \sim \beta} \Pr[B_\alpha \wedge B_\beta]$$

$$= \frac{1}{2}\binom{n}{3} 3(n-3)p^5 = \Theta(n^4 p^5)$$

- Only "second moment" calculation

For $p = o(n^{-4/5})$:

$$\Pr[\text{no triangles}] \sim e^{-\mu}$$

- Estimates very low probabilities

# No Copy of $H$

$X_H =$ number of copies of $H$

$$E[X_H] = \Theta(n^v p^e)$$

Janson, Łuczak, Rucinski:

$$\Pr[\text{no } H] = e^{-\Theta(\min E[X_{H'}])}$$

Example: For $n^{-2/3} \ll p < n^{-2/5}$

$$\Pr[\text{no } K_4] = e^{-\Theta(n^4 p^6)}$$

For $n^{-2/5} < p \ll 1$

$$\Pr[\text{no } K_4] = e^{-\Theta(n^2 p)}$$

EPIT: Every $v$ in Triangle

$$\binom{n-1}{2}p^3 = \mu$$

$$e^{-\mu} = \frac{c}{n}$$

JS: $\Pr[\text{EPIT}] \sim e^{-c}$

$X_v$ indicator r.v. for $v$ in no triangle, $X = \sum X_v$.

$$E[X_v] = \Pr[\wedge \overline{B_{vxy}}] \sim e^{-\mu} = \frac{c}{n}$$

$$E[X_{v_1} \cdots X_{v_r}] = \Pr[\wedge \overline{B_{v_i xy}}] \sim e^{-r\mu} = \left(\frac{c}{n}\right)^r$$

Inclusion-Exclusion:

$$\Pr[X = 0] \sim e^{-c}$$

$$x + y + z = n$$

$f_S(n) = \#x, y, z \in S, x + y + z = n$

Erdős, Tetali: There *exists* $S$

$$f_S(n) = \Theta(\ln n)$$

Pick $S$ random with

$$\Pr[x \in S] = p_x = c_1 \left(\frac{\ln x}{x^2}\right)^{1/3}$$

$$E[f_S(n)] = \sum_{x+y+z=n} p_x p_y p_z \sim k \ln n$$

Pick $c_1$ so $k > 50$

JANSON:

$\Pr[f_S(n) = 0] \sim \exp[-E[f_S(n)]] = n^{-k+o(1)}$

Janson Lower Tail:

$Pr[f_S(n) \leq \frac{k}{2} \ln n] < n^{-2}$

Janson Upper Tail:

$Pr[f_S(n) \geq \frac{3k}{2} \ln n] < n^{-2}$

Borel-Cantelli: $S$ almost surely works.

# (A) Proof of Janson Inequality

$$M \leq \Pr[\wedge \overline{B_\alpha}] \leq e^{-\mu + (\Delta/2)}$$

Lower: FKG

Upper: Order $B_1, \ldots, B_m$.

Bound $P_i^* := \Pr[B_i | \overline{B_1} \wedge \cdots \wedge \overline{B_{i-1}}]$

Renumber $i \sim 1, \ldots d$ of $1, \ldots, i-1$

$B := \overline{B_1} \wedge \cdots \wedge \overline{B_d}$; $C = \overline{B_{d+1}} \wedge \cdots \wedge \overline{B_{i-1}}$

$P_i^* = \Pr[B_i | B \wedge C] \geq \Pr[B_i | C] \Pr[B | B_i \wedge C]$

$P_i^* = \text{Pr}[B_i | B \wedge C] \geq \text{Pr}[B_i | C] \, \text{Pr}[B | B_i \wedge C]$

Independence: $\text{Pr}[B_i | C] = \text{Pr}[B_i]$

Incl-Excl: $\text{Pr}[B | B_i \wedge C] \geq 1 - \sum_{j=1}^{d} \text{Pr}[B_j | B_i \wedge C]$

FKG: $\text{Pr}[B_j | B_i \wedge C] \leq \text{Pr}[B_j | B_i]$

$P_i^* \geq \text{Pr}[B_i] - \sum_{j=1}^{d} \text{Pr}[B_j \wedge B_i]$

$1 - P_i^* \leq \exp[-\text{Pr}[B_i] + \sum_{j=1}^{d} \text{Pr}[B_j \wedge B_i]]$

Multiply over $i = 1, \ldots, m$:

$M \leq \exp[-\mu + (\Delta/2)]$

## Proof of Extended Janson Inequality

Assume Janson Inequality

$$\Pr[\wedge \overline{B_\alpha}] \le e^{-\mu + \Delta/2}$$

$$-\ln \Pr[\wedge_S \overline{B_\alpha}] \ge \sum_S \Pr[B_\alpha] - \frac{1}{2} \sum_S \Pr[B_\alpha \wedge B_\beta]$$

$S$ *random*, $\Pr[\alpha \in S] = p = \frac{\mu}{\Delta}$

$$E\left[-\ln \Pr[\wedge_S \overline{B_\alpha}]\right] \ge p\mu - p^2\Delta/2 = \frac{\mu^2}{2\Delta}$$

*Therefore there* exists $S$

$$\Pr[\wedge_S \overline{B_\alpha}] \le e^{-\mu^2/2\Delta}$$

*But*

$$\Pr[\wedge_{all} \overline{B_\alpha}] \le \Pr[\wedge_S \overline{B_\alpha}]$$

In mathematics whatever you learn is yours and you build it up – one step at a time. It's not like a real time game of winning and losing. You win if you are benefited from the power rigor and beauty of mathematics. It is a big win if you discover a new principle or solve a tough problem.

Fan Chung Graham