COCOON 2003

LIAR!

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Joint work with

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Paul versus Carole

N Possibilities

Q Yes/No Paul Queries

K (or fewer) Carole Lies

Try it with $N = 100$, $Q = 10$, $K = 1$

Carole plays Adversary Strategy

$\Rightarrow$ Perfect Information

$\Rightarrow$ Winning Strategy for Paul or Carole

$B_K(Q) =$ maximal N so that Paul Wins

Theorem:

$$B_K(Q) \sim \frac{2^Q}{\binom{Q}{K}}$$

Carole Strategy

Notation

$$\binom{Q}{\leq K} = \sum_{I=0}^{K} \binom{Q}{I}$$

Theorem: $N\binom{Q}{\leq K} > 2^{Q} \Rightarrow$ Carole Wins

Proof 1: Preserve Ministrategies

Proof 2: Random Play

Proof 1 $\Rightarrow$ Proof 2: Derandomization

Vector Format

Position $(3, 14)$ $((x_0, \ldots, x_K))$

Paul Move $(1, 9)$ $((a_0, \ldots, a_K))$

Yes: $(1, 11)$; No: $(2, 6)$

Perfect Split: Yes=No

Position $(8, 20)$, Move $(4, 10)$, Yes/No $(4, 14)$

$L : (x, y) \to (\frac{x}{2}, \frac{x}{2} + \frac{y}{2})$ $(L : R^{K+1} \to R^{K+1})$

Position after perfect split.

Problem: Integrality

Weight Function $W_Q(\vec{x}) = L^Q(\vec{x}) \cdot \vec{1}$

$W_Q(x, y) = 2^{-Q}((Q + 1)x + y)$

$(2^{-Q}(\binom{Q}{\leq K} x_0 + \ldots + (Q + 1)x_{K-1} + x_K))$

Paul Strategy

Theorem (JS): (K fixed, Q large)

$W \leq 1$ *and* $> cQ^K$ "pennies"

$\Rightarrow$ Paul Win

Keep Weight Equal (Perfect Split if Possible)

$Q = 10$. Position $(17, 837)$. $W = 1$

Paul $(8, 418 + x) \Rightarrow (8, 427 + x); (9, 427 - x)$

$W_9(1, -2x) = 0 \Rightarrow x = 5$

Problem: Nonnegativity

Proof Outline

First K Moves: Initial Penny Supply

Middle: Pennies Replenished from Nonpennies

End: Endgame Analysis

Halflie: No False Negatives

N Possibilities

Q Queries

K Halflies

$A_K(Q) =$ maximal N, Paul Wins

Theorem (Cicalese/Mundici, COCOON00):

$A_1(Q) \sim 2^{Q+1}/Q$

Dumitriu/JS:

$$A_K(Q) \sim 2^K B_K(Q) \sim 2^K \frac{2^Q}{\binom{Q}{K}}$$

Position $\vec{x} = (x, y)$ $((x_0, \ldots, x_K))$

Paul Query: (a, b) $((a_0, \ldots, a_K))$

Yes $(a, b + x - a)$; No $(x - a, y - b)$

Perfect Split $(\frac{x}{2}, \frac{y}{2} - \frac{x}{4})$

Yes/No $L\vec{x} := (\frac{x}{2}, \frac{y}{2} + \frac{x}{4})$ $(L : R^{K+1} \rightarrow R^{K+1})$

Problems: Integrality, Nonnegativity

Weight $W_Q(\vec{x}) = L^Q(\vec{x}) \cdot \vec{1}$

$W_Q(x, y) = 2^{-Q}(x(1 + \frac{Q}{2}) + y)$

$2^{-Q}(x_0 p_K(Q) + \ldots + x_{K-1}(1 + \frac{Q}{2}) + x_K)$

Paul Strategy

Start $(N, 0)$, $N < (1-\epsilon)2^{Q+1}/Q$

- Give Ground to (N, N)

$T := \lfloor \lg N \rfloor$

- Roundoff so $2^T | N$

- T perfect splits to $L^T(N\vec{1})$

- Endgame: Win in R from

$(0, 2^R)$; $(1, 2^R - 1)$; $(2, 2^R - 3)$; $(3, 2^R - 5)$

A Combinatorial Approach

1-Set: Subset of $\{Y, N\}^Q$ with

stem	$YNNYNY$
child	$Y\underline{Y}YNNY$
child	$YN\underline{Y}YYN$
child	$YNNY\underline{Y}N$

0-Set: Any Singleton

K-Set: Depth K tree with marked "lies."

parent	$Y\underline{Y}YNNYN$
child	$Y\underline{Y}YN\underline{Y}NN$
grandchild	$Y\underline{Y}YN\underline{YY}Y$

Theorem: Paul Wins from $(x_0, \ldots, x_K)$ in Q

$\Leftrightarrow$ Can Pack x_i $k - i$-Sets in $\{Y, N\}^Q$

Bound Packing of K-Sets

- When all have $\geq L$ Y, Size $> \binom{L}{\leq K}$

$L \sim \frac{Q}{2}$ Volume Bound $2^Q / \binom{Q/2}{K}$

$o(2^Q Q^{-K})$ have any $L < (1 - o(1))\frac{Q}{2}$

$A_K(Q) < (1 + o(1))2^Q / \binom{Q/2}{K}$

Careful Cutoff

Set $L = \frac{Q}{2} + c\sqrt{Q}\sqrt{\ln Q}$ Y

$$A_K(Q) \leq \frac{2^Q}{\binom{Q/2}{K}}(1 + cQ^{-1/2}\sqrt{\ln Q})$$

Yan/JS: Remove $\sqrt{\ln Q}$

Second Order Terms

Theorem (Yan/JS):

$$A_K(Q) = \frac{2^Q}{\binom{Q/2}{K}}(1 + \Theta(Q^{-1/2}))$$

Paul Strategy for $K = 1$

$W = 1 + cQ^{-1/2}$ at start.

Perfect Splits until $R \sim \frac{Q}{10}$ queries remain

New position (x, y) with

$$W = 2^{-R}[x(1 + \frac{R}{2}) + y] \sim 1 + c'R^{-1/2}$$

$$2^{-R}x(1 + \frac{R}{2}) \sim \epsilon$$

Find x disjoint “small” 1-Sets

Disjoint Shadows

A K-shadow consists of stem $\sigma \in \{Y, N\}^R$ and all τ derived by changing at most K Y to N. K-shadows are K-sets.

$K = 1$: Stems $(\epsilon_1, \ldots, \epsilon_R)$ with

$$\sum_{\epsilon_i = Y} i \equiv c \bmod R + 1$$

have disjoint 1-Shadows

- All stems with $\leq \frac{R}{2} - \sqrt{R}$ Y

- Average c

More than x disjoint 1-shadows, all small.

$$x\frac{R}{2} + y \sim 2^R(1 + c'R^{-1/2})$$

$$x\frac{R}{2} \sim \epsilon 2^R$$

$$x(\frac{R}{2} - \sqrt{R}) + y < 2^R$$

- Find x small disjoint 1-Shadows

- Add y 0-Sets as filler.

Arbitrary Channel

T-ary queries

E lie patterns

Example with $T = 3$, $E = 4$

Ternary Answers A/B/C.

Carole may lie B for A, A for B, A or B for C.

Theorem (Dumitriu, JS):

$$A_K^*(Q) \sim \frac{T^K}{E^K} \frac{T^Q}{\binom{Q}{K}}$$

Open Question

What is the maximum number $G(R)$ of disjoint 1-Shadows in $\{Y,N\}^R$?

$$\frac{2^R}{R+1} \le G(R)$$

$$G(R) \le 2\frac{2^R}{R}(1+o(1))$$

Asymptotic Factor of Two Gap.