1 The Rich Get Richer

Consider two bins, each of which initially have one ball. At each time $u = 1, 2, \ldots$ we add one ball to one of the bins. The ball is placed randomly, in proportion to the *square* of the number of balls already in the bin. (E. g. if the bins have 5 balls and 3 balls respectively the next ball is placed in the bin with 5 balls with probability $\frac{25}{25+9}$.)

Theorem 1.1 *With probability one, one of the bins will get all but a finite number of the balls.*

We move to a continuous time model. Let X_i be independent random variables, X_i having the exponential distribution with mean i^{-2}. (That is, X_i has density function $i^2 e^{-ti^2}$ for $t \geq 0$.) At time zero the first bin has one ball. It receives its second ball at time X_1. In general, it receives its i-th ball time X_i after receiving its $i-1$-st ball. Let X_i' also be independent exponential distributions with mean i^{-2}, independently chosen from the X_i. The second bin receives its balls according to the X_i'. The process ends when an infinite number of balls have been placed. The fictitious continuation, of defining the X_i, X_i' for all $i \geq 1$, shall be helpful in the analysis.

We use two basic properties of exponential distributions. Both are easy Calculus exercises.

• Let X be exponential with mean μ and let $a > 0$. Then $X - a$, conditional on $X \geq a$, is also exponential with mean μ. This is often called the *forgetfulness* property.

• Let X, X' be independent exponentials with means μ, ν respectively. Then $\Pr[\min(X, X') = X] = \frac{\mu^{-1}}{\mu^{-1}+\nu^{-1}}$.

The continuous time process mirrors the sequential process. Clearly the first ball is equally likely to go into either of the two bins. Suppose at some time $t > 0$ the first (say) bin has just received its i-th ball and the second bin last received its j-th ball at time $t' < t$. (When the second bin has not yet received its second ball set $j = 1$ and $t' = 0$.) The waiting time for the first bin is then X_i. The waiting time for the second was X_j at time t'. By the forgetfulness property its conditional waiting time at time t is X_j^*, exponential with mean j^{-2}. The next ball goes into the first bin if and only if $\min(X_i, X_j^*) = X_i$ which occurs with probability $i^2/(i^2 + j^2)$ as desired.

Let $T = \sum_{i=1}^{\infty} X_i$, $T' = \sum_{i=1}^{\infty} X_i'$ be the total times for the bins to receive (under fictitious continuation) an infinite number of balls. As $E[X_i] = E[X_i'] = i^{-2}$ and (critically!) $\sum_{i=1}^{\infty} i^{-2}$ converges, both T, T' have finite means and so are finite with probability one. As sums of independent continuous distributions $\Pr[T = T'] = 0$. Suppose $T < T'$, the other case being identical. At time T the first bin has received an infinite number of balls. The second bin has not. Therefore, the second bin has received only a finite number of balls!