Suppose $p >^* p_{k-1,s}$, all s. Fix an arbitrary tree T with any $k-1$ specified vertices $v_1, \ldots, v_{k-1}$ and with any specified integers $d_1, \ldots, d_{k-1}$ with each d_i at least the degree of its respective v_i in T . Then a.s. there will be contained in $G(n,p)$ an induced copy of T with the vertex corresponding to v_i having degree precisely d_i. In the countable models there will be trees having $k-1$ (or less) vertices of finite degree forming all possible finite subtrees.

When $p >^* p_{k-1,s}$, all s, and $p <^* p_{k+1,0}$ the countable models of $G(n,p)$ are distinguished by the trees containing precisely k vertices of finite degree. Let T be a finite tree on, say, m vertices with distinguished vertices $y_1, \ldots, y_k$. Suppose further that the y_i include all the leaves of T. Let $l_1, \ldots, l_k$ denote the degrees of $y_1, \ldots, y_k$ respectively in T. Let $d_1, \ldots, d_k \geq 0$. Let A be the event that $G(n,p)$ contains an induced copy of T and that y_i has degree precisely $l_i + d_i$. For this A we set $s = m + d_1 + \ldots + d_k - k \geq 0$ and

$$p_A = \frac{\log n}{kn} + \frac{(k-s+1)\log\log n}{kn}$$

Then if $p <^* p_A$, $\neg A$ holds a.s. while if $p >^* p_A$ then A holds a.s..

Suppose $p_{k,s-1} <^* p <^* p_{k,s}$. Then the countable model of $G(n,p)$ is determined: those tree components with precisely k points of finite degree exist if and only if the points and their degrees match the criteria above. Conversely, for each such T, $y_1, \ldots, y_k$, $d_1, \ldots, d_k$ meeting the criteria there will be countably many such components.

range $p = \Theta(\frac{\log n}{n})$ the threshold functions are "tighter" and there is still room for p to satisfy the Zero-One Law. The crucial sentences (for which we gratefully acknowledge the assistance of N. Pippenger) are the following, defined for $k \geq 1, s \geq 0$

• $A_{k,s}$: There exist $x_1, \ldots, x_k$ forming a path of length k, x_1 only adjacent to x_2, x_i only adjacent to x_{i-1}, x_{i+1} for $1 < i < k$, and x_k adjacent only to x_{k-1} and precisely s other vertices $y_1, \ldots, y_s$.

The special case $k = 1$ simplifies to

• $A_{0,s}$: There is a vertex of degree precisely s.

Set

$$p = p_{k,s}(n) = \frac{\log n}{kn} + \frac{(k+s-1)\log\log n}{kn} + \frac{c}{kn}$$

There are $\sim n^{k+s}/s!$ potential $x_1, \ldots, x_k, y_1, \ldots, y_s$ and each satisfies the condition with probability $\sim p^{k+s-1}(1-p)^{kn}$. With this p the expected number of such sets is then $e^{-c}/s!$ and

$$\Pr[A_{u,s}] \to 1 - e^{-e^{-c}/s!}$$

For notational convenience write $p(n) <^* q(n)$ if $n(q(n) - p(n)) \to \infty$ and $p(n) >^* q(n)$ if $n(q(n) - p(n)) \to -\infty$

Theorem 4. $p = n^{-1+o(1)}$ satisfies the Zero-One Law if and only if

$$p << n^{-1} \text{ or } p >> n^{-1}$$

and for all $k \geq 1, s \geq 0$

$$p <^* p_{k,s} \text{ or } p >^* p_{k,s}$$

The other cases having been handled in [SS] we may assume $p = \Theta(\frac{\log n}{n})$. In [SS] it has been noticed that for every m a.s. there do not exist m vertices with $m+1$ (or more) edges. For every $m \geq 3$, r there a.s. do exist (at least) r cycles of size precisely m. For every m, s a.s. every m vertices that have m edges have each vertex of degree (at least) s. Thus countable models of $G(n,p)$ would consist only of trees and unicylic components. The unicyclic components are determined: for each $m \geq 3$ there will be countably many components with a single cycle of length m and all degrees infinite. The distinctions come in the tree components.

Suppose $p <^* p_{k+1,0}$. For all m, r a.s. there do not exist m vertices joined in a tree containing $k+1$ vertices of degree at most r. In the countable models no tree can contain $k+1$ (or more) vertices of finite degree.

By induction on i we define $a_1, \ldots, a_i$ and sets $E_0 \supset E_1 \supset \ldots E_i$, all infinite. With E_{i-1} having been defined split $a \in E_{i-1}$ into two classes according to whether A_i or $\neg A_i$ holds a.s. with $p = n^{-a}$. As all a are irrational this gives a strict dichotomy. Let E_i be the infinite class, or either class if both are infinite. (This step is nonrecursive. Even for $i = 1$ there is no decision procedure that determines if A holds a.s. with $p = n^{-a}$ for infinitely many $a \in E_0$.) For notational convenience let B_i denote either A_i or $\neg A_i$, whichever gave the class E_i. Select $a_i \in E_i$, $a_i < a_{i-1}$, arbitrarily. Note that as the E_i are a descending sequence we have that for all $i \leq j$ that B_i holds a.s. in $G(n, n^{-a_j})$. For each j we may therefore pick an n_j so that for $n \geq n_j$

$$\Pr[G(n, n^{-a_j}) \models B_i] \geq 1 - \frac{1}{j}, 1 \leq i \leq j$$

Replacing n_j by $\max(n_1, \ldots, n_j)$ we can further assure $n_1 \leq n_2 \leq \ldots$. Now we define $p = p(n)$ by letting $p(n)$ be arbitrary for $n < n_1$ and setting

$$p(n) = n^{-a_j}, n_j \leq n < n_{j+1}$$

As $a_j \to 1/7$, $p(n) = n^{-1/7+o(1)}$. As all $a_j > 1/7$, $p(n) < n^{-1/7}$. For each i, we have that for each $j \geq i$, $\Pr[B_i] \geq 1 - \frac{1}{j}$ for $n_j \leq n$ and so B_i holds a.s. and therefore A_i holds with probability approaching either zero or one.

3 $p = n^{-1+o(1)}$

In this section we assume $p = n^{-1+o(1)}$ throughout and we characterize those p that satisfy the Zero-One Law. In [SS] it is shown that if

$$p << n^{-1}$$

or

$$n^{-1} << p << n^{-1}(\log n)$$

or

$$n^{-1}(\log n) << p$$

then p does satisfy the Zero-One Law. When $p = c/n$ the probability that $G = G(n, p)$ is trianglefree approaches $e^{-c^3/6}$. Hence for p to satisfy the Zero-One Law we must have $p << n^{-1}$ or $p >> n^{-1}$. When $p = \frac{\log n}{n} + \frac{c}{n}$ the probability that $G = G(n, p)$ has no isolated points is $e^{-e^{-c}}$. (This is better known as the threshold function for connectivity.) However, in the

Proof. Say $\mathcal{T}$ has ternary predicates $R_1, R_2, \ldots$. In the theory of graphs make a sentence A that there exist $x_1, \ldots, x_7$ so that $S = N(x_1, \ldots, x_7)$ has size $k(n)$ and on S we have a model of $\mathcal{T}$. We do this by replacing each second order quantified ternary $\exists_{R_i}$ by $\exists_{v_4, v_5, v_6, w_1, \ldots, w_{500}}$ and replacing $R_i(v_1, v_2, v_3)$ by $\vee_{i=1}^{500} N(v_1, v_2, v_3, v_4, v_5, v_6, w_i) \neq \emptyset$. Then a.s. A holds if and only if $k(n) \in S$. (Strictly speaking these ternary relations would be symmetric and hold for three unequal arguments. The somewhat technical modification to handle quantification over all ternary relations is discussed in [SS]. Binary and unary relations are handled similarly. Of course, the symbols used for $v_4, \ldots, w_{500}$ must be different in each replacement.) With $p = p(n)$ satisfying the Zero-One Law we must have $k(n) \in S$ being either true for all sufficiently large n or false for all sufficiently large n and as $k(n) \to \infty$ this implies Fact 9 .

This Fact gives a great strengthening of the results of [SS]. For example, let $\beta < \epsilon_0$ and let f_β denote the β-th function in the transfinite Ackermann heirarchy.

Fact 10 . If $\kappa(n) > f_\beta^{-1}(n)$ for all sufficiently large n then $p = p(n)$ does not satisfy the Zero-One Law.

Proof. As $k(n) \sim \kappa(n)^{1/3}$ this would imply $k(n) > f_{\beta+1}^{-1}(n)$. For $k \in K$ let k^+ denote the next element of K in ascending order. Say $k = k(n)$. Then $k[f_{\beta+1}(n)] > k$ and

$$k^+ \leq k[f_{\beta+1}(n)] \leq f_{\beta+1}(n) \leq f_{\beta+1}(f_{\beta+1}(k)) \leq f_{\beta+2}(k)$$

But with ternary predicates we may simulate arithmetic and the set S of those k with $f_{\beta+2}^{-1}(k)$ even is a spectrum. Since in K, k^+ is so "near" k it can't "jump over" the interval $[f_{\beta+2}(s), f_{\beta+2}(s+1))$ and so both $S \cap K$ and $\overline{S} \cap K$ would be infinite.

A. Blass (Ann Arbor) has shown that there is no recursive set K that meets the conditions of Fact 9. If $p = p(n)$ were given by a recursive function then the set K derived from it would be recursive and so by Blass's result p would not satisfy the Zero-One Law. The conditions on p seem almost contradictory. But they're not:

Theorem 3 . There exists a function $p = p(n)$ with $p < n^{-1/7}$ and $p = n^{-1/7+o(1)}$ satisfying the Zero-One Law.

Proof. Order the sentences in the first order theory of graphs $A_1, A_2, \ldots$. Set

$$\alpha_i = \frac{1}{7} + \frac{\sqrt{2}}{i}$$

or any sequence of irrational numbers decreasing to $1/7$. Set $E_0 = \{\alpha_1, \alpha_2, \ldots\}$.

• For every set S of size at most $10l$ and every 6-graph $\mathcal{H}$ on S with at most $l/10$ hyperedges there is a $w \notin S$ so that for all $v_1, \ldots, v_6 \in S$

$$\{v_1, \ldots, v_6\} \in \mathcal{H} \longleftrightarrow N(v_1, \ldots, v_6, w) \neq \emptyset$$

Extending and limiting this: for every 2-graph H on S with at most $50l$ edges there exist $v_3, v_4, v_5, v_6, w_1, \ldots, w_{500}$ so that for all $v_1, v_2 \in S$

$$\{v_1, v_2\} \in H \longleftrightarrow \vee_{i=1}^{500} N(v_1, \ldots, v_6, w_i) \neq \emptyset$$

We now say that a set S is bigger than a set S' if for some $v_3, v_4, v_5, v_6, w_1, \ldots, w_{500}$ the H thus defined on $S \cup S'$ gives an injection which is not a bijection from $S' - S$ to $S - S'$. When $|S \cup S'| < 5l$ this has the meaning of bigger. We can now say $MAX(x_1, \ldots, x_7)$, that $N(x_1, \ldots, x_7)$ has maximal size (as all possible sets N have size less than $2l$). We may say S has size $i(\text{mod} 10)$, that there is a graph H on S which is the union of 10-cliques plus i more points. For $0 \leq i < 10$ let A_i be the sentence that there exist $x_1, \ldots, x_7$ for which $N(x_1, \ldots, x_7)$ has maximal size and its size is $i(\text{mod} 10)$. Then $A_0 \vee \ldots \vee A_9$ holds a.s. so if p satisfies the Zero-One Law precisely one A_i holds a.s. This implies there must be a $k'(n)$ with $l - 4 \leq k'(n) \leq l + 4$ so that

$$\max |N(x_1, \ldots, x_7)| = k'(n)$$

a.s. Now set

$$k(n) = \lfloor (k'(n))^{1/3} \rfloor$$

We may say that a set $S = N(x_1, \ldots, x_7)$ has size $k(n)$: it has maximal size so that there exist S_1, S_2 of the same size, all disjoint, and an injection from $S \times S_1 \times S_2$ into a set T of size $k'(n)$. Now $k(n)^3 \leq k'(n) \leq l + 4 \leq 50l$. Any 3-graph H on S has less than $50l$ hyperedges so there exist $v_4, v_5, v_6, w_1, \ldots, w_5 00$ so that for all $v_1, v_2, v_3 \in S$

$$\{v_1, v_2, v_3\} \in H \leftrightarrow \vee_{i=1}^{500} N(v_1, v_2, v_3, v_4, v_5, v_6, w_i) \neq \emptyset$$

Now let K denote the set of values $k(n)$. A function $p = p(n)$ satisfying the Zero-One Law will determine the set K, up to the finite segment. Now let $\mathcal{T}$ be any *second* order sentence with quantification over unary, binary and ternary predicates as well as normal first order quantification. Set $S = Spec(\mathcal{T})$, i.e., the set of m for which there is a model of $\mathcal{T}$ containing exactly m elements.

Fact 9 . If $p = p(n)$ satisfies the Zero-One Law then for any such S either $K \cap S$ or $K \cap \overline{S}$ must be finite.

Some calculation ($s = r$ being the main term) gives that if

$$p = [7(\log n + \omega \log\log n)/n]^{1/7}$$

and $\omega = \omega(n) \to \infty$ then $\Pr[\neg A_r] \to 0$ for all r. We prove in §2 that these A_r give the "final threshold functions" with $p = n^{-1/7+o(1)}$. That is, if p is this large but still $p = n^{-1/7+o(1)}$ then p satisfies the Zero-One Law. Actually we state our result for any rational $\alpha = a/b \in (0,1)$. There remains a small gap in this aspect of our characterization of those $p = n^{-1/7+o(1)}$, $p > n^{-1/7}$ for which the Zero-One Law holds. The result in [SS] gives, for example, that if $p < n^{-1/7}(\log n)^{1/7-\epsilon}$ then the Zero-One Law does not hold. If $p > c(\log n)^{1/7} n^{-1/7}$, $c > 7^{1/7}$, then the Zero-One Law holds. The precise characterization of those p in this gap for which the Zero-One Law holds we do not here explore.

In §5 we extend the results of [SS] for $p < n^{-1/7}$ and show that p must satisfy a very severe restriction in order to satisfy the Zero-One Law. In particular, we show that there does not exist a *recursive* function $p = p(n)$ of this form. Nonetheless, we prove that there do exist $p = n^{-1/7+o(1)}$ with $p < n^{-1/7}$ that do satisfy the Zero-One Law.

2 Approaching $n^{-1/7}$

Here we ask: what $p = p(n)$ with $p = n^{-1/7+o(1)}$ and $p < n^{-1/7}$ satisfy the Zero-One Law. We shall show that the restrictions on such p are very severe. We shall also show that there are p with that property. While we consider only the exponent $-1/7$ the results may be extended to any rational exponent $\alpha \in (0,1)$. We write

$$p(n) = n^{-\frac{1}{7} - \frac{1}{\kappa(n)}}$$

and assume $\kappa(n) \to \infty$. We make heavy use of the results of [SS, §3]. In particular, we may assume

$$\kappa(n) < \log\log\log\log\log(n)$$

as otherwise we know p does not satisfy the Zero-One Law. Let $N(x_1, \ldots, x_7)$ denote the set of neighbors of $x_1, \ldots, x_7$. Let $l = l(n) = \lfloor \kappa(n) \rfloor$. The following hold a.s. in $G(n,p)$.

- For every $0 \le i \le l-4$ there exist $x_1, \ldots, x_7$ with precisely i neighbors.
- $l - 4 \le \max |N(x_1, \ldots, x_7)| \le l + 4$

For such p to satisfy the Zero-One Law we must have

$$p = n^{-\alpha+o(1)}$$

for some α. For otherwise we would have $p = n^{-\beta+o(1)}$ on one subsequence and $p = n^{-\gamma+o(1)}$ on another with $1 \geq \beta > \gamma \geq 0$. There would be a rational α strictly between β and γ and then the sentence $G(n,p) \supset H_\alpha$ would have probabilities approaching zero and one on the respective subsequences. When α is *irrational* we showed in [SS] that any $p = n^{-\alpha+o(1)}$ does satisfy the Zero-One Law. The situation with $\alpha = 1$ will be treated in §6. When $\alpha = 0$ the classic results of [F], [GKLT] give that if $p > n^{-\epsilon}$ for all positive ϵ and $1 - p > n^{-\epsilon}$ for all positive ϵ then the Zero-One Law is satisfied. For p so close to 1 that the second condition is not satisfied we reduce to $p = o(1)$ by noting, interchanging adjacency with nonadjacency, that p satisfies the Zero-One Law if and only if $1 - p$ does.

This leaves us with the central object of this paper: $p = n^{-\alpha+o(1)}$ where α is a rational number between zero and one. As $p = n^{-\alpha}$ is itself a threshold function, we split the possible p into two categories:

$$p >> n^{-\alpha} \text{ and } p = n^{-\alpha+o(1)}$$

and

$$p << n^{-\alpha} \text{ and } p = n^{-\alpha+o(1)}$$

In [SS] it is shown how to represent fragments of arithmetic in $G(n,p)$ for p near $n^{-1/7}$. In particular, the following result holds.
Theorem 1. There is a first order B such that for any $p = (q/n)^{1/7}$ with

$$n^{-1/\log\log\log\log\log n} < q(n) < \frac{\log n}{\log\log\log\log\log n}$$

$\Pr[G(n,p) \models B]$ does not approach a limit in n.

The constant $1/7$ in this theorem can be replaced by any rational $\alpha \in (0,1)$. The statement of this theorem reveals a surprising asymmetry when p is near $n^{-1/7}$. It seems that the Zero-One Law breaks down in the wide range before p achieves $n^{-1/7}$ but is reestablished not very far after it. Let A_r be the first order sentence that every seven vertices have at least r neighbors. If $\neg A_r$ then there exist $x_1, \ldots, x_7$ and $y_1, \ldots, y_s$ with $s \leq r$ with all y_j adjacent to all x_i and no other z adjacent to all x_i. We bound $\Pr[\neg A_r]$ by the expected number of such configurations.

$$\Pr[\neg A_r] \leq \sum_{s=0}^{r} \binom{n}{7}\binom{n}{s} p^{7s}(1-p^7)^{n-7-s}$$

When Does the Zero-One Law Hold?

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1 Introduction

Let $G(n,p)$ be a random graph with vertices $[n] = \{1,\ldots,n\}$, each pair of which appears as an edge of $G(n,p)$ independently with probability p. We shall say that a function $p = p(n)$ *satisfies the Zero-One Law* if for all statements A of the first order theory of graphs (whose precise definition we postpone until the beginning of the next section)

$$\lim_{n\to\infty} \Pr[G(n,p) \models A] = 0 \text{ or } 1$$

The proof that any constant p satisfies the Zero-One Law was given by Fagin[F] and Glebskii et. al. [GKLT]. Shelah and Spencer [SS] first considered $p = p(n)$ which varies as a function of n. Our results in this work are a natural continuation of [SS]. In particular, we are motivated by the following question: Which functions $p = p(n)$ satisfy the Zero-One Law?

Let α be 2; $1 + \frac{1}{k}$ for $k = 1, 2, \ldots$; 1; or any positive *rational* number between 0 and 1. Then $p = n^{-\alpha}$ does not satisfy the Zero-One Law. Indeed, there is a finite graph $H = H_\alpha$ so that $p = n^{-\alpha}$ is the threshold function for containment of a subgraph H. If $p(n) << n^{-\alpha}$ (i.e. $n^\alpha p(n) \to 0$) then $G(n,p)$ a.s. does not contain H while if $p(n) >> n^{-\alpha}$ (i.e. $n^\alpha p(n) \to \infty$) then $G(n,p)$ a.s. does contain H. Moreover, if $p(n) \sim cn^{-\alpha}$ for any fixed $c > 0$ then $\Pr[G(n,p) \supset H]$ approaches a limit between zero and one. (For example, when $p = cn^{-2/3}$, the probability that $G \supset K_4$ approaches $e^{-c^6/24}$.) For $p = p(n)$ to satisfy the Zero-One Law we must have $p(n) << n^{-\alpha}$ or $p(n) >> n^{-\alpha}$ for all such α. In [SS] we showed that this condition was also sufficient in the "very sparse" case: If $p << n^{-2}$ or $n^{-1-\frac{1}{k}} << p << n^{-1-\frac{1}{k+1}}$ for some $k = 1, 2, \ldots$ then $p = p(n)$ does satisfy the Zero-One Law. Henceforth we restrict ourselves to $p > n^{-1+o(1)}$.