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• There are no x, y with $y = x + r$ and $U(x), U(y)$
A countable model may be given on $Z \times Z$. There is a natural lexicographical $<$ with $(x, y) = \alpha < \beta = (x', y')$ if and only if $x < x'$ or $x = x' \wedge y < y'$. Define the clockwise ternary relation $C(\alpha, \beta, \gamma)$ by $\alpha < \beta < \gamma$ or $\beta < \gamma < \alpha$ or $\gamma < \alpha < \beta$. Now erase the $<$. Let $U(x, y)$ hold if and only if $y = 0$. (In our minds eye we have a countable number of lines and U holds for precisely one point on each line.)

When $n^{-1/2} \ll p \ll n^{-1/3}$ the situation gets more complex. Now pairs of close elements have appeared but triples have not. Here is one axiom schema.

• There are no x, y, z with $U(x), U(y), U(z)$ and all pairs less than r apart.
• If $U(x), U(y), U(x'), U(y')$ and x, x' are within r and y, y' are within r then there are (at least) s points z between y and x' with $U(z)$.
• (for any $a_1, \ldots, a_m < r$) There are $x_1, \ldots, x_m$ in order with $U(x_i)$ and $U(x_i + a_i)$, all i and no other z with $C(x_1, z, x_m)$ and $U(z)$ and $U(z + c)$ for any $c \leq r$.

A countable model can be given on $Z \times Z \times Z$. Again order it lexicographically, use the order to define C, and erase the order. Let $a_x, x \in Z$ be a doubly infinite sequence of positive integers so that every finite sequence of positive integers appears as a subsequence of consecutive values of a_x. Now for each x we define U on $\{x\} \times Z \times Z$ by $U(x, y, 0)$ for all $y \in Z$ and $U(x, 0, a_x)$. Here the picture is of a countable number of planes, stacked into a 3-space. Each plane has a countable number of lines, one of which has two points on it. The situation with even larger values of k seems to get even more complex but it appears that for the range $n^{-1/(k-1)} \ll p \ll n^{-1/k}$ there is a countable model on Z^k. It would be interesting to prove, for example, that there is no such model on Z^{k-1}.

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so that M may be regarded as a property of unary predicates U. Basically (see, e.g., [14]) the power of finite state machines M is the power of the language of unary predicates with $<$ plus the power of being able to count occurances $\bmod m$ for some fixed m. Using this one can show that for every machine M there exists m so that the following holds for all k : Let $p = p(n)$ satisfy $n^{-1/(k+1)} \ll p(n) \ll n^{-1/k}$. Fix $0 \leq j < m$. Let $n \to \infty$ so that $n \equiv j \bmod m$. Let $\epsilon_1 \ldots, \epsilon_n$ be a random sequence with $\Pr[\epsilon_i = 1] = p$, these events mutually independent. Then the probability that M accepts $\epsilon_1 \ldots, \epsilon_n$ approaches a limit, independent of the particular choice of $p = p(n)$.

12 Complete Theories and Countable Models

Any Zero-One Law leads to a complete theory - simply the set of sentences A that hold almost surely. And any complete theory has countable models. The new Zero-One laws with $p = p(n)$ sparse lead to a wealth of new complete theories and new problems on countable models. The case of $G(n, p)$ with $p = n^{-\alpha}$, $\alpha \in (0, 1)$ irrational has been examined in [23,24]. Suppose, for partial definiteness, that $.5 < \alpha < .51$. Only some of the extension sentences will be in the complete theory. The complete theory will include

$$\forall_x \exists_y x \sim y \tag{35}$$

and

$$\forall_{w,z} \exists_{x,y} w \sim x \wedge x \sim y \wedge y \sim z \tag{36}$$

but not, as mentioned earlier, that every two vertices have a common neighbor. The complete theory has been axiomatized in terms of those extension sentences it does have and certain other nonexistence sentences and a schema of generic extension. These theories are different for different α so that there are a *continuum* of distinct complete theories. The theories are not $\aleph_0$- categorical but there is a particularly interesting countable model. This is a minimal model that satisfies all the extension statements in the theory. One creates the model step by step. If, for example, 36 is an axiom and vertices $37, 92$ have already been created then at some time two new vertices, say $683, 684$ are created with $37, 683, 684, 92$ forming a path and no other edges from $x < 683$ to either 683 or 684.

The situation with $U^o(n, p)$ is also intriguing. When $n^{-1} \ll p \ll n^{-1/2}$ the complete theory can be axiomatized by the following schema. (r ranges over all positive integers.)

- There are r distinct x with $U(x)$

exists x so that the restriction of the graph to the set of z adjacent to x has property A. If A holds for a finite H add a focal point to H giving H^+; when H^+ occurs as a component then A^+ holds. Thus A^+ would hold almost surely. The Traktenbrot theorem gives Nonseparability.

Let A have quantifier depth t. Let $H_1, \ldots, H_s$ be a list of representatives for the Ehrenfeucht classes of depth t that have a finite representative and let H consist of t disjoint copies of each H_i. Almost surely G_n contains H as an isolated graph. All such graphs G_n have the same Ehrenfeucht value and so A holds either almost always or almost never.

11 Sparse Unary Predicates

The model $U_<(n,p)$ can profitably be considered with $p = p(n)$. Consider the property $\exists_x U(x)$. Its immediate to show that $p = n^{-1}$ is a threshold function. The sentence (using the conventions of §2)

$$\exists_x \exists_y y = x + 1 \wedge U(x) \wedge U(y) \tag{33}$$

has the meaning that some two consecutive elements have U and this property can be shown to have threshold function $n^{-1/2}$. More generally $n^{-1/k}$ is the threshold function for there being k consecutive elements with U. In work in progress [2] it is shown, roughly, that these are the only threshold functions. (We assume $p \leq \frac{1}{2}$ as p and $1-p$ will have the same properties.) More precisely, for any k it is shown that if $n^{-1/(k+1)} \ll p(n) \ll n^{-1/k}$ then Convergence holds and furthermore the value $\lim_{n\to\infty} \Pr[U_<(n,p) \models A]$ does not depend on the particular $p(n)$ but only on A and k.

Dolan [7] has shown that the Zero-One law holds if and only if $p \ll n^{-1}$ or $n^{-1} \ll p \ll n^{-1/2}$. If $n^{-1/2} \ll p \leq \frac{1}{2}$ he noted that (using the conventions of §2)

$$\exists_x U(x) \wedge U(x+1) \wedge \neg U(x+2) \wedge \neg \exists_y \left[U(y) \wedge U(y+2) \wedge \neg U(y+1) \wedge y < x \right] \tag{34}$$

has limiting probability $\frac{1}{2}$. One may consider this example a little artificial as it depends on the edges of $\{1, \ldots, n\}$. The circular $U^o(n,p)$ effectively removes those edge effects and, indeed, when $n^{-1/(k+1)} \ll p(n) \ll n^{-1/k}$ the random $U^o(n,p)$ does indeed satisfy the Zero-One law.

These investigations have an interesting application to Finite State Machines. Such a machine M either accepts or rejects a string $\epsilon_1 \ldots, \epsilon_n$ of bits. Such a string may naturally be regarded as a unary predicate on $\{1, \ldots, n\}$

cutting near $\frac{2n}{3}$ one gets some $y_{l'} \oplus \ldots \oplus y_{s+1} = s$ which is usually persistent. Then P has value $p \oplus m \oplus s = p \oplus s$. When an element is dropped in the middle it usually does not change the waists in the first or last third and so gives P' on $n+1$ elements with class $p \oplus m' \oplus s = p \oplus s$. The probability of any of the above not occuring can be shown to drop exponentially. As before, the values $a_n = \Pr[P(n,p) \models A]$ satisfy $|a_{n+1} - a_n| < c^{-n}$ and therefore converge.

The classification of random posets $P(n,p)$ with $p = p(n) \rightarrow 0$ remains an open and intriguing problem.

10 Convergence and Nonseparability

Our usual experience is that when Convergence is proven an algorithm can be given that finds $\lim_{n\rightarrow\infty} \Pr[R_n \models A]$ within any prescribed positive ϵ. Compton [3] has shown this need not be the case. Indeed, there are some quite natural examples where this is very much not the case.

Fix a constant p and consider the random poset $P(n,p)$. In the above section we indicated an argument of Łuczak that Convergence holds. Nonseparability is quite simple. For any sentence A in the first order theory of partial orders let A^* be the sentence that there exist x, y so that the partial order restricted to those z with $x \leq z \leq y$ has property A. Any finite partial order almost surely appears as the restriction of $P(n,p)$ to some interval. If A holds for any finite partial order then A^* holds almost surely, otherwise A^* holds almost never. Using Traktenbrot's theorem one can show there is no decision procedure for determining if A holds for some finite partial order, and hence Nonseparability holds for $P(n,p)$.

Let $f(d)$ be a function defined on the positive integers with all $0 \leq f(d) \leq 1$. Define a random graph G_n on vertex set $\{1,\ldots,n\}$ by letting i,j be adjacent with probability $f(d)$, where $d = |i-j|$, each adjacency determined independently. Varying the function f gives an interesting class of random structures. Let us take $f(d) = 2^{-d}$ to be specific. The language is simply the first order language of graphs. In particular, $<$ is not in the language. This random structure satisfies the Zero-One Law and Nonseparability.

The key probability result is that every finite graph H almost surely appears as an isolated graph in G_n. That is, there are no adjacencies between the vertices v in the copy of H and those not in the copy. Indeed, this property alone implies the Zero-One Law and Nonseparability.

Given any sentence A about graphs let A^* be the sentence that there

The above are all equivalent. Furthermore, if $x, y \in M$ are persistent then

$$\forall_m x \oplus m \oplus y = x \oplus y \tag{32}$$

Roughly, a U in class x is persistent if it has all "nonedge" properties up to quantifier depth t. For example, it must have five (for appropriate t) consecutive x with $U(x)$. However, $U(1)$ may or may not hold.

For U on $\{1, \ldots, n\}$ let U^L, U^R denote the restriction of U to the first and last $\frac{n}{3}$ values respectively. With U selected from distribution $U_<(n,p)$ the values U^L, U^R are independently selected from distribution $U_<(\frac{n}{3}, p)$. The value of the equivalence class of U^L (and similarly U^R) is then the result of a random walk of length $\frac{n}{3}$ on directed graph G^R in which one begins at $\emptyset$ (the equivalence class of the null sequence) and moves from x to $x \oplus i$ with $i = 0, 1$ chosen with probabilities $1-p, p$ respectively. The probability of avoiding a minimal closed set in this graph of fixed size drops exponentially with time and so at time $\frac{n}{3}$ is less than c^{-n} for some $c < 1$.

We may think of $U_<(n+1,p)$ as being generated from $U_<(n,p)$ by dropping in a new element in the middle at position $\frac{n}{2}$. With probability at least $1 - 2c^{-n}$ we may write the class of $U_<(n,p)$ as $x \oplus m \oplus y$, representing the first, middle and last third of the sequence, with x, y persistent. Dropping in an extra element in the middle changes m to m' but by 32 this will not affect the value. Any A of quantifier depth t may be regarded as the union of classes in M. From this, the values $a_n = \Pr[U_<(n,p) \models A]$ can change by only $O(c^{-n})$ when $U_<(n,p)$ is changed to $U_<(n+1,p)$. As $\sum |a_{n+1} - a_n|$ is finite, the a_n must approach a limit.

A similar argument has been given by Luczak [15] to show that the random poset $P(n,p)$ satisfies Convergence for any constant p. Fix t and let M be the set of equivalence classes of posets under the t-move Ehrenfeucht game. Define the sum $P = P_1 \oplus P_2$ of two posets (letting V_1, V_2 denote their underlying sets, assumed disjoint) by letting $x < y$ and $P(x,y)$ have their induced meanings if $x, y \in V_1$ or $x, y \in V_2$ and setting $x < y$ for all $x \in V_1, y \in V_2$ and, critically, setting $P(x,y)$ for all $x \in V_1, y \in V_2$. As before, this induces an addition $\oplus$ on M.

On $P(n,p)$ call x a *waist* if $y < x \rightarrow P(y,x)$ and $x < y \rightarrow P(x,y)$. That is, in the underlying G there are increasing paths from x to $x+1, x+2, \ldots$ and decreasing paths from x to $x-1, x-2, \ldots$. The probability x is a waist is larger than a positive constant α. $P(n,p)$ usually has waists $x_1, \ldots, x_s$ with $s \sim \alpha n$. These waists split P into $P_1 \oplus \ldots P_s \oplus P_{s+1}$ with P_i the restriction of P to $[x_{i-1}, x_i - 1)$. Letting y_i be the class of P_i, P has class $y_1 \oplus \ldots \oplus y_{s+1}$. Cutting near $\frac{n}{3}$ one gets some $y_1 \oplus \ldots \oplus y_l = p$ which is usually persistent and will win $EHR(S^1, S^2, t)$. Let $n = m \rightarrow \infty$. With probability $\frac{1}{2}$ precisely one of independently chosen $S^1, S^2 \sim R_n$ have property A and so Spoiler's probability of winning $EHR(S^1, S^2, t)$ is at least $\frac{1}{2}$ and so 31 does not hold.

In general, the appeal of the Ehrenfeucht game approach to *non*logicians is that 31 is a statement involving "only" probability and discrete math and so is approachable without any background in Logic. A simple (to graph theorists) proof of the Glebskii/Fagin Theorem may be given using the Ehrenfeucht Game. Fix $p \in (0,1)$. The random $G \sim G(n,p)$ has what P. Winkler has dubbed the "Alice's Restaurant" property. For any fixed r, s almost surely for any distinct $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$ there exists a z adjacent to all of the x_i and none of the y_j. To win $EHR(G^1, G^2, t)$ Duplicator on each round needs only to find a vertex with the appropriate adjacencies to previously chosen vertices, and almost surely he can do so.

Let $p = n^{-\alpha}$ with $\alpha \in (0,1)$ irrational. Now the Alice's Restaurant property no longer holds. Suppose, for example, $.5 < \alpha < .51$. Then it almost surely is not true that every two vertices have a common neighbor. Here Duplicator's strategy must be more subtle. In [22] a winning Duplicator strategy is given which gives an alternate proof of the Zero-One law of [20].

9 Convergence via Ehrenfeucht

Here we give an argument somewhat different from [18] to show that $U_<(n,p)$ has convergence (p constant) and then show how this may be applied to $P(n,p)$, p constant. Consider models U of unary predicates with underlying $<$. Fix an integer t and define the $U \equiv U'$ if Duplicator wins $EHR(U, U', t)$. This is the equivalence relation of having the same quantifier depth t properties. Let M be the (finite!) set of equivalence classes. There is a natural addition $U_1 \oplus U_2$ given by placing the "sequence" U_2 "to the right of" U_1. This induces an addition $x \oplus y$ on M which gives a semigroup. Let $0, 1$ denote the equivalence classes of the models on $\{1\}$ with $\neg U(1)$, $U(1)$ respectively. On M define a directed graph G^R with edges $(x, x \oplus i)$, $x \in M$, $i = 0, 1$ and a directed graph G^L with edges $(x, i \oplus x)$, $x \in M$, $i = 0, 1$. We say x is *persistent* if

- x lies in a minimal closed set of G^R
- x lies in a minimal closed set of G^L
- $\forall_y \exists_z x \oplus y \oplus z = x$
- $\forall_y \exists_z z \oplus y \oplus x = x$
- $\exists_p \exists_s \forall_m p \oplus m \oplus s = x$

one sets $p_i(n) = i/\binom{n}{2}$ for $i = 1, 2, \ldots$. Set

$$\delta_{ij} = \lim_{n\to\infty} \Pr[G_<(n, p_i(n)) \models A_j] \tag{30}$$

where the A_j enumerate the sentences of the theory. One needs the existence of the limit, Convergence for $p_i(n)$. Given that one goes to a subsequence i_l for which the $\delta_{i_l,j}$ converge for all j and then one patches together these $p_{i_l}(n)$ into a $p(n)$ with the desired properties. The other, more complicated, portion of the argument requires an encoding of binary relations on sets whose size is a given function of $f(n)$.

8 The Ehrenfeucht Game

There are quite naturally several approaches to proving a Zero-One Law but one that is of particular interest to graph theorists is via the Ehrenfeucht game. Given two disjoint structures S^1, S^2 and a positive integer t the Ehrenfeucht game, lets call it $EHR(S^1, S^2, t)$ is a t-round perfect information game between two players, Spoiler and Duplicator. On each round Spoiler selects a vertex from either structure (his choice, and it may vary from turn to turn) and then Duplicator must select a vertex from the other structure. Let $x_1, \ldots, x_t$ and $y_1, \ldots, y_t$ denote the vertices selected from S^1, S^2 respectively, the index denoting the round. Duplicator wins if the structures resticted to these moves are isomorphic. In the case of graphs that requirement is that $x_i \sim x_j$ if and only if $y_i \sim y_j$. With ordered graphs we also require $x_i < x_j$ if and only if $y_i < y_j$ and in general all predicated in the underlying language must be preserved. The now classic result [9] is that S^1, S^2 are elementarily equivalent if and only if for every t Duplicator (with perfect play) wins $EHR(S^1, S^2, t)$

Theorem. The random structures R_n satisfy the Zero-One Law if and only if for every t

$$\lim_{n,m\to\infty} \Pr[\text{Duplicator wins } EHR(R_n, R_m, t)] = 1 \tag{31}$$

Here we think of structures S^1, S^2 being chosen independently from probability spaces R_n, R_m respectively. Again, both players are assumed to play perfectly.

To get a feeling for the argument imagine there is a sentence A for which $\lim_{n\to\infty} \Pr[R_n \models A] = \frac{1}{2}$. Let t be the quantifier depth of A. Basic analysis of the Ehrenfeucht game gives that if $S^1 \models A$ and $S^2 \models \neg A$ then Spoiler recursively defined. By itself, of course, this does not prove the nonexistence of recursive $p(n)$.

We outline the argument for (b) with $p = p(n)$ given by 22. Let $N = N(x, y, z)$ be the set of common neighbors of x, y, z and let $H(x, y, z)$ be the predicate that $N \neq \emptyset$. Set

$$q = \Pr[H] \sim np^3 \sim n^{-3/\kappa(n)} \tag{29}$$

For this outline we will regard regard H as a random predicate with probability q. Any choice of z induces a binary predicate $H(x, y)$. The maximal size $|N|$ is known to be $\sim \kappa$. With N^1, N^2 disjoint (for convenience) call N^1 bigger than N^2 if there is a z so that the binary H on $N^1 \cup N^2$ gives an injection which is not a bijection from N^2 to N^1. With sets of size, say, $\kappa/10$ each H has probability $\sim q^{\kappa/10} \gg n^{-1}$ of giving this injection (assuming it exists) and almost surely one of the n choices of z gives that H. Sets of common neighbors N with no bigger N^1 then must have size $\Theta(\kappa)$. By technical means one can now speak of sets of common neighbors N of size $\sim \kappa^*$ with $\kappa^* = \kappa(n)^{1/3}$. Now *any* symmetric binary R on N holds for at most $(\kappa^*)^2 \ll \kappa$ pairs and so H has probability at least $q^{o(\kappa)} \gg n^{-1}$ of being R. Almost surely the n choices of z induce all binary R on N.

To avoid technical difficulties, suppose we can speak of $N = N(x, y, z)$ having precisely $\kappa^*(n) = \lfloor \kappa(n)^{1/3} \rfloor$ elements. Let B be any sentence in the first order theory of graphs and let $S = Sp(B)$ be the (usual) spectrum of B, the set of m for which there is a graph on m vertices satisfying B. Let B^* be the first order sentence with the almost surely meaning that there exist x, y, z so that $N = N(x, y, z)$ has κ^* elements and there exists w so that the graph on N given by $H(-, -, w)$ satisfies B. Let T be the set of values $\kappa^*(n)$. For n large $\Pr[B^*]$ will be near one when $\kappa^*(n) \in S$ and near zero otherwise. If $\Pr[B^*] \to 1$ then all large $t \in T$ have $t \in S$ while if $\Pr[B^*] \to 0$ then all large $t \in T$ have $t \notin S$. For $p = p(n)$ to satisfy the Zero-One Law it must satisfy it for sentences B^* and hence either $T \cap S$ or $T \cap \overline{S}$ must be finite for every possible spectrum S. Results of Fagin [11] and others give that the class of such spectrum S is very broad so that, as argued in [16], no recursive T, hence no recursive $p(n)$, can have this property.

This phenomenon also appears in joint work, still in progress, of J. Lynch and P. Dolan on the random ordered graph $G_<(n, p)$. Consider the case $p = f(n)/\binom{n}{2}$ with $f(n) \to \infty$ but slowly, say, $f(n) < \ln n$. Basically these G look like $\sim f(n)$ disjoint edges in a scrambled order. There is such a $p(n)$ for which Convergence holds but there is no such recursive function. Here or, equivalently, that we may write

$$p = p(n) = n^{-\frac{1}{3} - \frac{1}{\kappa(n)}} \tag{22}$$

with $\kappa(n) \to +\infty$ and $\kappa(n) = o(\ln n)$.
Theorem (a) There exist $p = p(n)$ satisfying 21 for which the Zero-One law holds
(b) There are no *recursive* $p = p(n)$ satisfying 21 for which the Zero-One law holds.

We can give a fairly complete outline for (a). For $i = 1, 2, \ldots$ define

$$p_i(n) = n^{-\alpha_i} \text{ with } \alpha_i = -\frac{1}{3} - \frac{1}{\pi i} \tag{23}$$

Enumerate $A_1, A_2, \ldots$ the sentences of the first order theory of graphs. As α_i is irrational the Zero-One law holds for $p_i(n)$ and

$$\delta_{ij} = \lim_{n\to\infty} \Pr[G(n, p_i(n)) \models A_j] = 0 \text{ or } 1 \tag{24}$$

for all i, j. Standard diagonalization techniques give a subsequence $i_1 < i_2 < \ldots$ so that for each j there is a $\delta_j \in \{0, 1\}$ with

$$\delta_j = \delta_{i_j, j} = \delta_{i_{j+1}, j} = \delta_{i_{j+2}, j} = \ldots \tag{25}$$

Setting $\beta_l = \alpha_{i_l}$ for convenience, now define $p = p(n)$ by

$$p(n) = n^{-\beta_l} \text{ for } n_l \leq n < n_{l+1} \tag{26}$$

($p(n)$ arbitrary for $n < n_1$) where $n_1 < n_2 < \ldots$ are chosen so that

$$\left| \Pr[G(n, n^{-\beta_l}) \models A_j] - \delta_j \right| < \frac{1}{l} \tag{27}$$

for $1 \leq j \leq l$. Then for all j

$$\lim_{n\to\infty} \Pr[G(n, p(n)) \models A_j] = \delta_j \tag{28}$$

and so the Zero-One law holds.

Observe why this procedure is not itself recursive. Indeed, there is no recursive procedure to separate those A which hold almost always for $p_i(n)$ for i sufficiently large from those A which hold almost never for $p_i(n)$ for i sufficiently large. Hence the δ_j, and so the subsequence $i_1 < i_2 < \ldots$ is not it does so very slowly and since there are n choices for c almost surely one of them will work, and similarly there will be an $[e, f]$ inducing multiplication.) Call $[a, b]$ maximally arithmetizable if neither $[a-1, b]$ nor $[a, b+1]$ are arithmetizable. Almost surely all such maximally arithmetizable intervals will have $\alpha = \log\log n < b-a$ and, trivially, $b-a \leq n$, and furthermore there will be maximally arithmetizable intervals. With $[a, b]$ aritmetized standard techniques allow us to say many things about the length $b-a$. Let $\log^* x$ be the number of times one must iterate log, beginning at x, until the number falls below 1. Almost surely

$$\log^* n > \log^*(b-a) > \log^*(n) - 2 \tag{20}$$

for all maximally arithmetizable intervals $[a, b]$. Consider the sentence A that there exists a maximal arithmetizable $[a, b]$ with $\log^*(b-a)$ between 0 and 50 modulo 100. If, say, $\log^* n \equiv 25 \bmod 100$ then almost surely the maximal arithmetizable $[a, b]$ have $log^*(b-a)$ either $25, 24$ or $23 \bmod 100$ and A holds. If, say, $\log^* n \equiv 75 \bmod 100$ then almost surely all arithmetizable $[a, b]$ have $\log^*(b-a)$ either $75, 74$ or $73 \bmod 100$ and A will not hold.

A similar argument works with $G(n, p)$ for $p = n^{-1/7}$ is given in [20]. Instead of intervals $[a, b]$ we have the set V of common neighbors of $x_1, \ldots, x_7$. An exterior vertex w generates a 6-ary predicate on V by saying $R(v_1, \ldots, v_6)$ if $v_1, \ldots, v_6, w$ have a common neighbor. V is arithmetized if the predicate induces plus and times. (There is a technical problem in that as stated this is a symmetric predicate and one has to desymmetrize it.) One V is shorter than another V' if there are $w_1, \ldots, w_5$ so that the binary predicate on their union given by $w_1 \ldots, w_5, x, y$ having a common neighbor provides an injection which is not a bijection from V to V'. One needs some (not so easy) technical lemmas that all maximal arithmetizable V will have at least, say, $\log\log n$ vertices and then the remainder of the argument is as before.

7 No Recursive Convergence

In [16], while giving a near complete classification of those $p = p(n)$ for which the Zero-One law holds for $G(n, p)$, a peculiar phenomenon was discovered. Restrict (just to be specific) $p = p(n)$ so that

$$\begin{aligned} p(n) &\ll n^{-1/3} \\ p(n) &= n^{-1/3-o(1)} \end{aligned} \tag{21}$$

i.e., that v_i, v_j, w have no common neighbor is $(1-p^3)^{n-3} \sim e^{-1}$ so the chance of getting the particular pentagon with v_i, v_{i+1} adjacent is $\epsilon_2 = (e^{-1})^5(1-e^{-1})^5$. The total probability that $H_{x,y,z,w} \cong H$ is then more than a positive constant $\epsilon = \epsilon_1\epsilon_2$. Almost surely one of $\frac{n}{4}$ disjoint choices of x, y, z, w will give this H. Of course, the notions of independence must be fleshed out to make this a full argument.

Assume the Representation Theorem. For any sentence A we can construct the still first order

$$A^* : \exists_{x,y,z,w} H_{x,y,z,w} \models A \tag{18}$$

If A holds for no finite graph then A^* holds for no finite graph and so $\Pr[A^*] \to 0$ in any probability space. If A holds for some finite graph H then in $G(n,p)$ with $p = n^{-1/3}$ there almost surely are x, y, z, w with $H_{x,y,z,w} \cong H$ so that A^* would hold almost surely. A decision procedure that could separate the A^* holding almost never from those holding almost always would separate the A holding for some finite H from those holding for no finite H and this contradicts the Traktenbrot Theorem.

The situation is even easier if we take, for example, $G_<(n,p)$ with p constant. With $<$ a built in relation we can transform any A into

$$A^* : \exists_{a,b} G|_{[a,b]} \models A \tag{19}$$

For any finite ordered graph H there almost surely are a, b with $G|_{[a,b]} \cong H$ and the proof is as above.

6 Proving Nonconvergence

We outline the argument that $G_<(n,p)$ has nonconvergence for $p = \frac{1}{2}$, or any constant. We want to give an interval $[a,b]$ the structure of an initial fragment of arithmetic. Given other intervals $[c,d]$ and $[e,f]$ induce on $[a,b]$ the ternary $x + y = z$ by $x, y < z$ and the existence of $v \in [c,d]$ adjacent to x, y, z and no other points in $[a,b]$. (The case $x = y$ can be handled specially.) We similarly use $[e,f]$ to induce the ternary predicate $xy = z$. Consider a as "1". Say $[a,b]$ is arithmetizable if there exist c, d, e, f so that with these induced definitions plus and times satisfy the axioms for a fragment of arithmetic. Take $\alpha = \alpha(n) = \log\log n$. Almost surely *any* $[a,b]$ with $b - a \leq \alpha$ will be arithmetizable. (Rougly, take some $[c,d]$ of size around α^2. If $[c,d]$ has just the right adjacencies to $[a,b]$ then it will induce plus and the probability that this occurs is like $2^{-\alpha^3}$. While this goes to zero and

$$\lim_{n\to\infty} \Pr[\text{no isolated triangles}] = e^{-c^3 e^{-3c}/6} \tag{14}$$

Lynch [19] showed that the limit always existed and further, as a function of c, that it must have a form "similar" to the examples above.

In the negative direction this author [21] has shown that Nonseparability holds for the random graph $G(n,p)$ with $p = n^{-1/3}$ and this was extended [6] by his then student Peter Dolan to $p = n^{-\alpha}$ for any rational $\alpha \in (0,1)$. Let $0 < p < 1$ be arbitrary but fixed and let $G_<(n,p)$ be the random ordered graph of §2. Compton, Hanson and Shelah [4] have shown that Nonconvergence holds; they have given A for which $\lim_{n\to\infty} \Pr[G_<(n,p) \models A]$ does not exist.

Actually, in the two previous examples both Nonconvergence and Nonseparability hold. This is often (not always, see §10) the case though for many random structures proving Nonconvergence requires considerably more technical effort than proving Nonseparability.

5 Proving Nonseparability

The basic tool usually used in proving nonseparability is the Traktenbrot theorem [25]. This result (in somewhat limited form) states that there is no decision procedure to determine if a sentence A in the first order theory of graphs is satisfied for any finite graph. We outline how this is used in [21] to show Nonseparability for $G(n,p)$ with $p = n^{-1/3}$. We define $H_{xyzw} = (V,E)$ by the first order

$$a \in V \longleftrightarrow a \sim x \wedge a \sim y \wedge a \sim z \tag{15}$$

and

$$\{a,b\} \in E \longleftrightarrow a,b \in V \wedge \neg\exists_v v \sim a \wedge v \sim b \wedge v \sim w \tag{16}$$

That is, given a graph G and vertices x,y,z,w this defines a new graph H. The definitions are designed for the following.

Representation Theorem. Let H be any finite graph. Then almost surely

$$\exists_{x,y,z,w} H_{x,y,z,w} \cong H \tag{17}$$

We give a rough outline when H is a pentagon. Pick $x,y,z,w \in G(n,p)$ at random. Each $a \in G$ has probability $p^3 = n^{-1}$ of being in V so that $|V|$ has a Poisson distribution with mean one and with probability $\epsilon_1 = \frac{1}{e5!}$ the size $|V| = 5$. Given $V = \{v_1, \ldots, v_5\}$ the probability that $\{v_i, v_j\} \in E$,

4 Examples

The classic example for the Zero-One law, as we have already mentioned, is the random graph $G(n,p)$ where p is any constant. With Saharon Shelah [20] we showed that the Zero-One law also holds when $p = p(n) = n^{-\alpha}$ and $0 < \alpha < 1$ is irrational. This has a natural description in terms of threshold functions. Let us take, as an example, the first order property A that the graph contains a K_4, i.e., that there exist four distinct vertices with all six pairs adjacent. It is know, even from the original [8], that $p = n^{-2/3}$ is the threshold function for this property. When $p \ll n^{-2/3}$ almost surely $\neg A$ while with $p \gg n^{-2/3}$ almost surely A. In between the $\Pr[A]$ moves from zero to one - the exact result is that with $p = cn^{-2/3}$

$$\lim_{n\to\infty} \Pr[G(n, cn^{-2/3}) \text{ contains no } K_4] = e^{-c^6/24} \tag{11}$$

but the only important thing for us is that it is neither zero nor one. The rough notion is that *at* a threshold function the probability is moving between zero and one so that for a Zero-One law to hold $p = p(n)$ must be between the threshold functions - it must be a *dull* function of n. While this has strong intuitive appeal to graph theorists we must point out that the notion of threshold function does not apply to all first order properties A and so the feeling of Zero-One meaning "not a threshold function" must remain only a feeling.

There are, of course, functions of n other than $n^{-\alpha}$. In [16] a fairly complete characterization of those $p = p(n)$ for which the Zero-One law holds is given.

A classic example of Convergence was given by Ehrenfeucht. He (as attributed in [18]) showed that, for any constant p, Convergence held for $U_<(n,p)$ - i.e., that $\lim_{n\to\infty} \Pr[U_<(n,p) \models A]$ existed for all sentences A. Another example concerns the random function F_n. Here we have a function symbol f taken uniformly from the n^n functions $f : \{1,\ldots,n\} \to \{1,\ldots,n\}$. Equality is the only other predicate. For example,

$$\lim_{n\to\infty} \Pr[\neg\exists_x f(x) = x] = e^{-1} \tag{12}$$

Lynch [17] showed that the limit always exists and, moreover, that it must have a particular form. A final example concerns the random graph $G(n,p)$ with $p = \frac{c}{n}$, c a constant. There, for example,

$$\lim_{n\to\infty} \Pr[\text{no triangles}] = e^{-c^3/6} \tag{13}$$

$\{1, \ldots, n\}$ as forming a cycle with 1 directly following n so that $C(x, y, z)$ has the meaning that x, y, z lie in a clockwise direction. (Note that *if* we added a constant symbol this could be reduced to $U_<(n, p)$.) We write $y = x + 1$ for $\neg\exists_z C(x, z, y)$ and inductively we can write $y = x + r$ for any $r \in N$. We say $x_1, \ldots, x_r$ are *in order* if $C(x_i, x_j, x_k)$ for all $1 \leq i < j < k \leq r$.

Random Poset $P(n, p)$

The language consists of the built in $<$ and a partial order P. To generate P first generate the random graph $G \sim G(n, p)$. Set xPy if there exists a sequence (for arbitrary r, including $r = 1$) $x = x_0 < x_1 < \ldots < x_r = y$ with all $\{x_i, x_{i+1}\} \in G$. (In other words set xPy if $x < y$ and x, y are adjacent and then take the transitive closure.) Sample sentence:

$$\exists_x \exists_y x \neq y \wedge \neg\exists_z [P(z, x) \vee P(z, y)] \tag{7}$$

meaning there is more than one minimal element under the partial order P.

3 Four Outcomes

In the general situation there will be defined for every n a random structure R_n of "size" n and a language $\mathcal{A}$. The assumption that A is a sentence of the language $\mathcal{A}$ shall be assumed tacitly throughout. We say A holds *almost always* if

$$\lim_{n\to\infty} \Pr[R_n \models A] = 1 \tag{8}$$

and we say A holds *almost never* if $\neg A$ holds almost always. We consider four possible outcomes. On the positive side:

- **Zero-One Law.** Every sentence A holds almost always or almost never.
- **Convergence.** For every sentence A

$$\lim_{n\to\infty} \Pr[R_n \models A] \text{ exists} \tag{9}$$

On the negative side:

- **Nonseparability.** No recursive decision procedure separates those A holding almost surely from those A holding almost never.
- **Nonconvergence.** There exists A for which

$$\lim_{n\to\infty} \Pr[R_n \models A] \text{ does not exist} \tag{10}$$

Observe that Nonseparability is formally stronger than saying that there is no recursive decision procedure for determining $\lim_{n\to\infty} \Pr[R_n \models A]$.

Two recent surveys [5,26] of these problems from somewhat different perspectives are well worth reading.

2 Five Structures

All of these structures have parameters n and p. The underlying space is always $\{1,\ldots,n\}$. The value $p \in [0,1]$ represents a probability . In all cases it is of interest to characterize the possible outcomes (see §3) for various $p = p(n)$. All languages are assumed to be first order with equality.

Random Graph $G(n,p)$

One symmetric areflexive binary predicate adjacency, denoted $x \sim y$. Structures are graphs G on $\{1,\ldots,n\}$ with probabilities determined by $\Pr[i \sim j] = p$, these events mutually independent. Sample sentences:

$$\exists_x \exists_y \exists_z x \sim y \wedge x \sim z \wedge y \sim z \tag{3}$$

$$\forall_x \exists_y x \sim y \tag{4}$$

with meanings "there exists a triangle" and "there is no isolated vertex" respectively.

Random Ordered Graph $G_<(n,p)$

The structure is as above but the language also contains the built in binary $<$. Sample sentence:

$$\exists_x \exists_y x \sim y \wedge \neg\exists_z \left[z \neq x \wedge z \neq y \wedge (z < x \vee z < y)\right] \tag{5}$$

with the meaning that $1,2$ are adjacent. We write "1" for that x so that $\neg\exists_y y < x$, $y = x+1$ for $x < y \wedge \neg\exists_z [x < z \wedge z < y]$ and similarly $y = x + r$ for any $r \in N$.

Random Ordered Unary Predicate $U_<(n,p)$

Language consists of unary U and binary $<$. Structures are unary predicates U with probabilities determined by $\Pr[U(i)] = p$, these events mutually independent. $<$ is the built in less than on $\{1,\ldots,n\}$. The conventions for Random Ordered Graphs apply. Sample sentence:

$$\exists_{x,y,z} y = x+1 \wedge z = x+2 \wedge U(x) \wedge U(y) \wedge U(z) \tag{6}$$

meaning that some three consecutive elements have U.

Random Circular Unary Predicate $U^o(n,p)$

U is as above. Instead of $<$ we have the built in ternary relation $C(x,y,z)$ with the meaning that $x < y < z$ or $y < z < x$ or $z < x < y$. We think of

Zero-One Laws With Variable Probability

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1 Introduction

One of this author's favorite theorems has long been the Zero-One law discovered independently by Glebskii *et.al.* [12] and Ron Fagin [10]. Let A be any first order property of graphs and let $\mu_n(A)$ be the proportion of labelled graphs on n vertices for which A holds. Then

$$\lim_{n\to\infty} \mu_n(A) = 0 \text{ or } 1 \tag{1}$$

This result has inspired much work by logicians, generally in the direction of showing 1 for more powerful languages. Thus it is known [5] that 1 holds when A is a sentence in fixed point logic and it is known [13] that 1 does not always hold when A is a sentence is second order monadic logic. Here, however, we explore recent work in a totally different direction. Let $G(n,p)$ denote the random graph on n vertices with edge probability p. (In §2 we define the random structures we will deal with.) A property A is an event in the probability space and $\Pr[G(n,p) \models A]$ is well defined. When $p = \frac{1}{2}$, each labelled graph on n vertices has equal weight so that 1 may be rewritten

$$\lim_{n\to\infty} \Pr[G(n,p) \models A] = 0 \text{ or } 1 \tag{2}$$

Fagin's proof actually gives that 2 holds for any constant $0 < p < 1$.

To people who work in Random Graphs the cases p constant are only a small and relatively uninteresting part of the theory. Rather, they consider the edge probability p to be $p = p(n)$, a function of the number of vertices n. Take, for example, connectivity - though this is not a first order property. In their classic paper Paul Erdős and Alfred Rényi [8] showed that if $p(n) \ll \frac{\ln n}{n}$ then G almost surely is not connected while if $p(n) \gg \frac{\ln n}{n}$ then G almost surely is connected. The called the function $\frac{\ln n}{n}$ the threshold function for connectivity. Over the years analogous threshold functions have been found for many natural graphtheoretic properties and invariably they involve the edge probability p as a function of the number of vertices n. The new direction in the study of Zero-One laws is to consider random structures (Random Graphs the main but not the only example) with a probability parameter p and to examine for which $p = p(n)$ the Zero-One law, and the other possibilities to be described, apply.