No books or notes. Do all problems. Maxscore: 200 plus Bonus

1. (30) Set $\alpha = 2^{1/19}$ and set $\epsilon = e^{2\pi i/19}$. Set $K = Q(\alpha, \epsilon)$, $L = Q(\alpha)$, $M = Q(\epsilon)$.
   
   (a) (5) Find $[M : Q]$. You must give a short reason!
   (b) (5) Find $[L : Q]$. Your must give a short reason!
   (c) (5) Is $M : Q$ a normal extension? You must give a short reason!
   (d) (5) Is $L : Q$ a normal extension? You must give a short reason!
   (e) (5) Is $K : Q$ a normal extension? You must give a short reason!
   (f) (5) Is $K : M$ a normal extension? You must give a short reason!

2. (20) For $\gamma = x + iy \in Z[i]$, $\gamma \neq 0$, $x, y \in Z$, set $d(\gamma) = x^2 + y^2$. Let $\alpha, \beta \in Z[i]$ with $\beta \neq 0$. Prove that there exist $\kappa, \rho \in Z[i]$ with $\alpha = \beta \kappa + \rho$ and either $\rho = 0$ or $d(\rho) < d(\beta)$. (A picture will help!)

3. (15) Let $p(x) \in Q[x]$ be an irreducible cubic with three roots $\alpha, \beta, \gamma$. Set $\kappa = \alpha^5 + \beta^5 + \gamma$.
   
   (a) (5) Show $[Q(\kappa) : Q] \leq 6$
   (b) (10) Show $[Q(\kappa) : Q] \neq 6$

4. (15) Let $K : F$ be a normal extension and let $G = \Gamma(K : F)$.
   
   (a) (5) For $H$ a subgroup of $G$ define $H^\dagger$.
   (b) (5) For $L$ a field with $F \subseteq L \subseteq K$ define $L^*$.
   (c) (5) Let $L = H^\dagger$, $a = [L : F]$, $b = |H|$, $c = |G|$. State (no proof requested!) a numerical relationship between $a, b, c$.

5. (10) Let $\alpha, \beta, \gamma$ denote the three roots of $x^3 + x^2 + x + 3 = 0$. Find $\alpha^2 + \beta^2 + \gamma^2$.

6. BONUS: Let $p$ be prime and let $f(x) \in Z_p[x]$ be an irreducible cubic. Let $a, b, c, d \in Z_p$ such that there does not exist $s \in Z_p$ with $as^3 + bs^2 + cs + d = 0$. Prove there exists a $g(x) \in Z_p[x]$ of degree one or two so that $f(x)$ divides $ag(x)^3 + bg(x)^2 + cg(x) + d$ in $Z_p[x]$. 
7. (20) Let \( p(x) = x^6 + ax^4 + bx^2 + c \in \mathbb{Q}[x] \). Assume \( p(x) \) is irreducible over \( \mathbb{Q} \). Let \( K = \mathbb{Q}(\alpha, \beta, \gamma, \delta, \kappa, \lambda) \) where \( \alpha, \beta, \gamma, \delta, \kappa, \lambda \) are the roots of \( p(x) \). Prove that \( \Gamma(K : \mathbb{Q}) \) is not isomorphic to \( S_6 \). Give an upper bound on \( |\Gamma(K : \mathbb{Q})| \).

8. (15) Let \( Q = L_0 \subset L_1 \subset \ldots \subset L_r = K \). Assume that for every \( 0 \leq i < r \) either \( L_{i+1} = L_i(\sqrt{\alpha}) \) with \( \alpha \in L_i \) or \( L_{i+1} = L_i(\alpha^{1/3}) \) with \( \alpha \in L_i \). Prove that \( 2^{1/5} \not\in K \).

9. (15) Let \( K = \mathbb{Z}_7[y]/(y^2 + 2y + 5) \). By “find” below we mean in the form \( a + by \) with \( a, b \in \{0, 1, 2, 3, 4, 5, 6\} \).
   (a) (5) Find \( (y + 3)(y + 4) \).
   (b) (5) Find \( \frac{y+3}{y+4} \).
   (c) (5) Find \( y^{9603} \).

10. (15) Let \( f(x), g(x) \) be monic polynomials with integer coefficients. Let \( p \) be a prime number. Let \( \lambda \) be a complex number. Assume that \( f(x) \) is the minimal polynomial of \( \lambda \) and that \( g(x) \) is the minimal polynomial of \( \lambda^p \).
   (a) (5) Prove that \( f(x) \) divides \( g(x^p) \) in \( \mathbb{Z}[x] \).
   (b) (10) Prove that \( f(x), g(x) \) have a nontrivial common factor when considered in \( \mathbb{Z}_p[x] \).

11. (10) Draw the roots of the equation \( z^4 + z^8 = -1 \) in the Complex Plane. [If you feel like bursting into song, please don’t!]

12. (10) How many \( x \in \mathbb{Z}_{701} \) satisfy \( x^7 = 1 \). You must give a reason for your answer. (Note: 701 is prime.)

13. (15) Let \( \alpha_1, \ldots, \alpha_r \in F \) be distinct and nonzero. Prove that the sequences \( \alpha_i^n \) are linearly independent.

14. (10) Let \( f(x) \in \mathbb{Z}_2[x] \) be irreducible of degree \( n \). Set \( K = \mathbb{Z}_2[x]/(f(x)) \).
   (a) (5) How many elements does \( K \) have?
   (b) (5) Show that \( x^{2^n-1} = 1 \) in \( K \).

It is simplicity that is difficult to make. – Bertolt Brecht