1. (15) Let $H$ be a Normal subgroup of $G$ and let $g \in G$. Define $Hg$. Define $gH$. Prove that $Hg = gH$.
Solution: We define
$$Hg = \{hg : h \in H\}$$
$$gH = \{gh : h \in H\}$$
Suppose $x \in Hg$. So $x = h_1g$ for some $h_1 \in H$. Then
$$x = (gg^{-1})h_1g = g(g^{-1}h_1g)$$
But as $H$ is normal, $g^{-1}h_1g \in H$ so $x \in gH$. That is, $Hg \subseteq gH$. The other direction, $gH \subseteq Hg$, is similar (we omit it here) and we conclude the sets are equal.

2. (15) Let $G$ be an Abelian group under multiplication and define
$$H = \{x \in G : x^5 = e\}$$
Prove that $H$ is a subgroup of $G$. In your proof, point out where the assumption that $G$ was Abelian was used.
Solution: As usual, three parts.
Identity: As $e^5 = e$, $e \in H$.
Product: Assume $x, y \in H$. Then $x^5 = y^5 = e$. Then
$$(xy)^5 = xyxyxyxyxy = x^5y^5 = ee = e$$
where here we used that $G$ was Abelian. So $xy \in H$.
Inverse: Assume $x \in H$. Then $x^5 = e$. Then
$$(x^{-1})^5 = (x^5)^{-1} = e^{-1} = e$$
so $x^{-1} \in H$.

3. (20) We define a group Tiger with elements eenie, meenie, minie, moe.
(You do not need to show this is a group.)
(a) (3) What is the identity of Tiger?
Solution: meenie as it times anything is that other thing.

(b) (2) Find the order moe.
Solution: \( \text{moe}^2 = \text{eenie}, \text{moe}^3 = \text{enniemoe} = \text{minie}, \text{moe}^4 = \text{moeminie} = \text{meenie}, \) the identity, so the order is four.

(c) (3) Find the inverse of minie.
Solution: moe

(d) (2) Is Tiger Abelian?
Solution: Yup. The table is symmetric.

(e) (10) Give a well known group that Tiger is isomorphic to and give the bijection between the elements of that group and eenie, meenie, minie, moe.
Solution: As moe has order four and there are four elements Tiger is isomorphic to \((Z_4, +)\), setting \(\phi(0) = \text{meenie}\) (the identity), \(\phi(1) = \text{moe}, \phi(2) = \text{moe}^2 = \text{eenie} \) and \(\phi(3) = \text{moe}^3 = \text{minie}\).

4. (15) Let \(\phi : G \to H\) be a homomorphism. Define \(W\) by

\[W = \{g \in G : \phi(g) = e\}\]

(a) (2) What word is commonly used for \(W\)?
Solution: Kernel

(b) (9) Prove that \(W\) is a subgroup of \(G\).
Solution: Again three parts:
Identity: As \(\phi(e) = e, e \in W\).
Product: Assume \(x, y \in W\). Then \(\phi(x) = \phi(y) = e\). Then
\[\phi(xy) = \phi(x)\phi(y) = ee = e\]
So \(xy \in W\).
Inverse: Assume \(x \in W\). Then \(\phi(x) = e\). Then
\[\phi(x^{-1}) = \phi(x)^{-1} = e^{-1} = e\]
so that \(x^{-1} \in W\).

(c) (4) Prove that \(W\) is a normal subgroup of \(G\).
Solution: Let \(w \in W\) and \(g \in G\). Then
\[\phi(g^{-1}wg) = \phi(g)^{-1}\phi(w)\phi(g) = \phi(g)^{-1}e\phi(g) = \phi(g)^{-1}\phi(g) = e\]
so that \(g^{-1}wg \in W\).
5. (10) Find \( o(3) \) in \( \mathbb{Z}_{13}^\ast \).
   **Solution:** \( 3^2 = 9, \ 3^3 = 27 = 1 \) so \( o(3) = 3 \).

6. (20) Let \( G = \mathbb{Z}^2 \) and let
   \[ H = \{(x, y) \in G : x + 2y \text{ is divisible by } 3\} \]
   Note: You can assume without proof that \( H \) is a subgroup of \( G \).
   (a) (5) Draw a picture of the portion of \( G \) with both coordinates between \(-4\) and \(+4\), marking those points which are in \( H \).
   **Solution:** Here \( O \) is the origin, which is in \( H \), other points of \( H \) marked by \( x \):
   ooxooxoox
   oooxoxxox
   xooxoxoox
   oxxoooxoox
   oxxoOooxo
   xoxxooxo
   ooxoxoox
   xxooxo
   oooxo
   oxx
   xox
   (b) (5) Find a nice set of representatives for \( G/H \).
   **Solution:** \((0, 0), (0, 1), (0, 2)\) as you get a point of \( H \) every third point going horizontal.
   (c) (5) Give the table for \( G/H \).
   \[
   \begin{array}{c|ccc}
   \cdot & (0, 0) & (0, 1) & (0, 2) \\
   \hline
   (0, 0) & (0, 0) & (0, 1) & (0, 2) \\
   (0, 1) & (0, 1) & (0, 2) & (0, 0) \\
   (0, 2) & (0, 2) & (0, 0) & (0, 1) \\
   \end{array}
   \]
   (d) (5) What well known group is \( G/H \) isomorphic to.
   **Solution:** This is \((\mathbb{Z}_3, +)\) as the second coordinate adds mod 3 and with these representatives the first coordinate stays zero.

7. (15) Let \( G \) be a finite Abelian group with \( n \) elements where \( n \) is not divisible by seven. Define \( \phi : G \to G \) by \( \phi(x) = x^7 \). You may assume, without proof, that \( \phi \) is a homomorphism. **Prove** that \( \phi \) is an isomorphism. (For partial credit state what you need to show.)
   **Solution:** Since \( G \) is finite we need only prove injectivity or surjectivity. Here we'll do injectivity, for which it suffices to show that the equation \( \phi(x) = e \) has only the solution \( x = e \). OK, so assume \( \phi(x) = x^7 = e \). Then \( x \) has either order seven or order one. But it can’t have order 7 since 7 does not divide the number of elements of the group. So it must have order one, that is, it must be the identity.