ALGEBRA SAMPLE MIDTERM

No books or notes.
There are a total of 100 points. Do all problems.
A Reminder $H$ is a normal subgroup of $G$ if it is a subgroup of $G$ and it has the further property that: for every $g \in G$ and $h \in H$, $ghg^{-1} \in H$.

1. (15) Let $H$ be a Normal subgroup of $G$ and let $g \in G$. Define $gH$. Prove that $Hg = gH$.
Solution: Let $x \in gH$. So $x = gh$ for some $h \in H$. So $x = gh(g^{-1}g) = (ghg^{-1})g$. By Normality, $ghg^{-1} \in H$ so $x \in Hg$. Conversely, let $x \in Hg$. So $x = hg$ for some $h \in H$. So $x = (gg^{-1})hg = g(g^{-1}hg)$. By Normality $g^{-1}hg \in H$ so $x \in gH$.

2. (15) Let $G$ be an Abelian group under multiplication and define $H = \{x \in G : x^5 = e\}$
Prove that $H$ is a subgroup of $G$. In your proof, point out where the assumption that $G$ was Abelian was used.
Solution:
Identity: $e^5 = e$ so $e \in H$.
Product: Suppose $x, y \in H$ so $x^5 = y^5 = e$. Then $(xy)^5 = x^5y^5 = ee = e$ so $xy \in H$. (Abelian used to say $(xy)^5 = x^5y^5$.)
Inverse: Suppose $x \in H$ so $x^5 = e$. Then $(x^{-1})^5 = (x^5)^{-1} = e^{-1} = e$ so $x^{-1} \in H$.

3. (20) We define a group Tiger with elements enie, meenie, minie, moe.
(You do not need to show this is a group.)

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(a) (3) What is the identity of Tiger?
Solution: meenie as it times everything is that thing.
(b) (2) Find the order \( \text{moe} \).
Solution: \( \text{moe}^2 = \text{eenie}, \text{moe}^3 = \text{eenie \cdot moe} = \text{minie}, \text{moe}^4 = \text{minie \cdot moe} = \text{meenie} \), the identity, so the order is 4.

(c) (3) Find the inverse of \( \text{minie} \).
Solution: \( \text{moe} \), as \( \text{minie \cdot moe} \) is the identity \( \text{meenie} \).

(d) (2) Is \( \text{Tiger} \) Abelian?
Solution: Yes, the table is symmetric.

(e) (10) Give a well known group that \( \text{Tiger} \) is isomorphic to and give the bijection between the elements of that group and \( \text{eenie}, \text{meenie}, \text{minie}, \text{moe} \).
Solution: \((\mathbb{Z}_4, +)\), identifying 0, 1, 2, 3 with \( \text{meenie}, \text{moe}, \text{eenie}, \text{minie} \) respectively. (Or identify with \( \text{meenie}, \text{minie}, \text{eenie}, \text{moe} \) respectively.)
4. (15) Let $\phi : G \to H$ be a homomorphism. Define $W$ by

$$W = \{ g \in G : \phi(g) = e \}$$

(a) (2) What word is commonly used for $W$?
Solution: Kernel

(b) (9) Prove that $W$ is a subgroup of $G$.
Solution:
  
  **Identity:** $\phi(e) = e$ so $e \in W$.
  
  **Product:** Suppose $x, y \in W$ so $\phi(x) = \phi(y) = e$. Then $\phi(xy) = \phi(x)\phi(y) = ee = e$ so $xy \in W$.
  
  **Inverse:** Suppose $x \in W$ so $\phi(x) = e$. Then $\phi(x^{-1}) = \phi(x)^{-1} = e^{-1} = e$ so $e \in W$.

(c) (4) Prove that $W$ is a normal subgroup of $G$.
Solution: Let $g \in G, w \in W$, so $\phi(w) = e$. Then

$$\phi(gwg^{-1}) = \phi(g)\phi(w)\phi(g^{-1}) = \phi(g)e\phi(g^{-1}) = \phi(g)\phi(g)^{-1} = e$$

so $gwg^{-1} \in W$.

5. (10) In $S_9$ find $o(\tau)$ where $\tau = \left( \begin{array}{c} 123456789 \\ 358914627 \end{array} \right)$

Solution: $o(\tau) = 20$ as $\tau$ cycles $1 \to 3 \to 8 \to 2 \to 5 \to 1$ and $4 \to 9 \to 7 \to 6 \to 4$ so the first cycle repeats in 5 and the second in 4 and so they simultaneously repeat in 20.

6. (20) Let $G = \mathbb{Z}^2$ and let

$$H = \{ (x, y) \in G : x + 2y \text{ is divisible by } 3 \}$$

Note: You can assume without proof that $H$ is a subgroup of $G$.

(a) (5) Draw a picture of the portion of $G$ with both coordinates between $-4$ and $+4$, marking those points which are in $H$.
Solution: O is origin, o marks other points of $H$.

(b) (5) Find a nice set of representatives for $G/H$.
Solution: $(0,0), (0,1), (0,2)$ as $H, H$ moved one to right, and $H$ moved two to right give everything.

(c) (5) Give the table for $G/H$.
Solution: Elements are $(0,0), (0,1), (0,2)$. Table
(d) (5) What well known group is $G/H$ isomorphic to.
Solution: $(Z_3, +)$, mapping $i$ to $(0, i)$.

7. (15) Let $G$ be a finite Abelian group with $n$ elements where $n$ is \textit{not} divisible by seven. Define $\phi : G \rightarrow G$ by $\phi(x) = x^7$. You may assume, without proof, that $\phi$ is a homomorphism. \textit{Prove} that $\phi$ is an isomorphism. (For partial credit state what you need to show.)
Solution: We first show $\phi$ is an injection. If $\phi(x) = e$ then $x^7 = e$ so the order of $x$ must be either 7 or 1. But for any $x \in G$, $o(x)|o(G) = n$ and as $n$ is not divisible by seven we must have $o(x) = 1$ so we must have $x = e$. Thus $\phi$ is injective. But an injective map from a finite set to a set of the same size (in this case from $G$ to itself) must also be surjective. So $\phi$ is bijective and hence an isomorphism.

‘A knot!’ said Alice, already to make herself useful, and looking anxiously about her. ‘Oh, do let me help to undo it!’
– from Alice in Wonderland, by Lewis Caroll