Algebra, Assignment 9
Due, Tuesday, Nov 10

1. Let $R$ be a ring. Call $a \in R$ a unit if $ab = 1$ for some $b \in R$. Let $X$ be the set of units. Prove that $X$ forms a group under multiplication. What is our standard notation for $X$ in the case where $R = \mathbb{Z}_n$?

2. Recall $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$. Find a unit (as defined above) $a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ which has $a \geq 10$. (One approach: Find some unit other than $\pm 1$ and apply previous problem.)

3. Let $\mathbb{Z}[i]$, as usual, denote the Gaussian Integers and set $I$ equal the multiples of $2 + i$. (That is, $I$ is all numbers $(2 + i)\alpha$ where $\alpha \in \mathbb{Z}[i]$.) Take a sheet of graph paper (or do it carefully by hand) and make $X$ and $Y$ axes. Now think of the point $(x, y)$ with $x, y \in \mathbb{Z}$ as corresponding to the Gaussian Integer $x + iy$. Mark the elements of $I$ by hand. (Well, $I$ is infinite of course but just mark those elements with both coordinates between $-10$ and $+10$. Make sure you get all of them. There should be a pleasing pattern.) Now consider the horizontal line $y = 2$. Looking at the pattern make a conjecture as to precisely which $x$ are marked: that is, for which $x$ is $x + 2i \in I$. Then, prove your conjecture. (A start: To determine whether or not $\beta \in I$ compute $\frac{\beta}{2+i}$ as a complex number. If this quotient is in $\mathbb{Z}[i]$ then $\beta \in I$, and if this quotient is not in $\mathbb{Z}[i]$ (that is, the real and/or complex coefficients are not integral) then $\beta \not\in I$.)

4. Continuing the above, for $\alpha \in \mathbb{Z}[i]$ let $\overline{\alpha}$ denote the coset in $\mathbb{Z}[i]/I$.
   (a) Set $\alpha = 37 + 22i$. Find an integer $n$ (note: by integer we will mean, as usual, an element of $\mathbb{Z}$, we shall be careful to use the term Gaussian Integer for elements of $\mathbb{Z}[i]$) so that $\overline{\alpha} = \overline{n}$. (Note: This means that $\alpha - n \in I$. Geometrically it means that there would be a marked point $n$ to the left of $\alpha$ on your graph paper, if it went that high.)
   (b) Find a positive integer $n$ as small as possible with $\overline{42} = \overline{n}$.
   (c) Find explicit $\alpha_1, \ldots, \alpha_s \in \mathbb{Z}[i]$ (you need find the right $s$) so that $\mathbb{Z}[i]/I = \{\overline{\alpha_1}, \ldots, \overline{\alpha_s}\}$
      (Note: To show this you need two things. First that the $\overline{\alpha_i}$ are distinct and second that for any $\beta \in \mathbb{Z}[i], \overline{\beta} = \overline{\alpha_i}$ for some $1 \leq i \leq s$.)
5. Let $R$ be a ring of characteristic 3. Prove that the map $\phi : R \to R$ given by $\phi(x) = x^3$ is a homomorphism.

Fearing error and fearing truth are one and the same. Those who fear making mistakes are incapable of discovery. When we worry about making mistakes, the error within us becomes an unmovable rock. In our fear, we cling to what we have declared to be “true” one day, or what has always been presented as such. When we are driven by a thirst for knowledge, and not by the fear of seeing a false security fade away, then error, like suffering and sadness, passes through us without ever gaining substance, and the trace it leaves is that of renewed knowledge.
Alexander Grothendieck, *Recôltes et Semailles*