1. Let $G$ be a finite group and let $a, b \in G$ with $a \sim b$. Show $o(a) = o(b)$.

2. Let $G$ denote the set of linear functions $f(x) = mx + b$ on the real line with $m \neq 0$. Denote such a function by $(m, b)$. Define a product $f^*g$ as the function $h(x) = g(f(x))$. (Check assignment 1 and the solutions for earlier work on this group.) Recall $C(f), N(f)$ denote the conjugate class and the normalizer (see §2.11) of $f$.

   (a) Describe $C(f)$ and $N(f)$ when $m \neq 1$ and $b \neq 0$.
   (b) Describe $C(f)$ and $N(f)$ when $m \neq 1$ and $b = 0$.
   (c) Describe $C(f)$ and $N(f)$ when $m = 1$ and $b \neq 0$.
   (d) Describe $C(f)$ and $N(f)$ when $m = 1$ and $b = 0$.
   (e) Describe $Z[G]$, the center of the group.

3. Let $\sigma \in S_n$ be (in cycle notation)

   $\sigma = (1\,2\cdots n)$

   (a) Describe in words the $\gamma \in C(\sigma)$.
   (b) Find $o(C(\sigma))$.
   (c) Deduce $o(N(\sigma))$.
   (d) (*) Describe $N(\sigma)$ explicitly. (Idea: Since you already know that $N(\sigma)$ has precisely woggle elements, you should look for woggle distinct elements that are in $N(\sigma)$ and then you have them all.)

4. Let $o(G) = p^n$, $p$ prime. Prove that $o(Z[G]) \neq p^{n-1}$. (Idea: Examine the proof that groups of order $p^2$ are Abelian.)

5. Let $C^*$ denote the nonzero complex numbers under multiplication.

   (a) Show that every $x + iy \in C^*$ can be uniquely expressed as

   $x + iy = r(\cos(\theta) + i\sin(\theta))$

   where $r$ is a positive real number and $0 \leq \theta < 2\pi$. 
(b) Suppose
\[ x_1 + iy_1 = r_1(\cos(\theta_1) + i \sin(\theta_1)) \]
and
\[ x_2 + iy_2 = r_2(\cos(\theta_2) + i \sin(\theta_2)) \]
Set \( x + iy = (x_1 + iy_1)(x_2 + iy_2) \) and write
\[ x + iy = r(\cos(\theta) + i \sin(\theta)) \]
Express \( r \) in terms of \( r_1, r_2 \) and \( \theta \) in terms of \( \theta_1, \theta_2 \).

(c) Use the above to argue that \( C^\ast \) is isomorphic to \( (R^+, \cdot) \times [(R, +)/(2\pi Z, +)] \)

Remark: The notation \( x + iy = re^{i\theta} \) is often used for this correspondence.

6. Show that \( Z_8^\ast \) is isomorphic to \( Z_2 \times Z_2 \) by exhibiting an explicit isomorphism. (Notation: \( Z_n^\ast \) always means under multiplication while \( Z_n \) means under addition.)

7. (*) Find an explicit isomorphism between \( Z_{15}^\ast \) and some product \( Z_a \times Z_b \).

The universe is not only queerer than we suppose but queerer than we can suppose.

– J.B.S. Haldane