ALGEBRA FINAL

Originality and a feeling of one’s own dignity are achieved only through work and struggle. – Dostoyevsky

No books or notes. Problems marked (-) are relatively easy while those marked (*) are particularly challenging. Maximal Grade 200.

DO THREE OF PROBLEMS 1,2,3,4

1. (15) Assume as a fact that when \( p \) is a prime \( \mathbb{Z}_p^* \) is cyclic. Assume as a fact that 77003 is prime. How many \( x \in \mathbb{Z}_{77003}^* \) satisfy \( x^7 = 2 \)? (You must give a clear reason for your answer!)

2. (15) Let \( G \) be a group and \( a \in G \). Define

\[
X(a) = \{ g \in G : ga = ag \}
\]

What is \( X(a) \) usually called? Prove that \( X(a) \) is a subgroup of \( G \).

3. (15) Give the definition of \( H \) being a normal subgroup of \( G \). (You can assume that subgroup has already been defined.) Is the group of \( n \times n \) real matrices of determinant one a normal subgroup of the group of \( n \times n \) nonsingular real matrices? Why?

4. (15) Let \( F, K \) be fields with \( F \subset K \) and \( [K : F] = 13 \). Let \( \alpha \in K \) with \( \alpha \notin K \). Show that \( \alpha \) satisfies a polynomial \( p(x) \in F[x] \) of degree 13. Further, show that \( \alpha \) satisfies a minimal polynomial \( p(x) \in F[x] \) of degree 13.

DO ALL PROBLEMS IN THIS SECTION

5. (25) Define the terms Integral Domain and Field. (You may assume the term Ring has been defined so you need only give the additional properties. Prove that a finite Integral Domain is a Field.

6. (20) Let \( G \) be an Abelian Group under multiplication and define \( \phi : G \to G \) by \( \phi(x) = x^3 \). Show that \( \phi \) is a homomorphism. Now assume further that \( G \) is a finite group and that \( o(G) \) is not divisible by 3. Show that \( \phi \) is a bijection.
7. (25) With $D_{10}$ as given in the attached sheet:
   
   (a) (-)(5) What is the inverse of $R2$?
   (b) (10) Find the conjugacy class of $R3$.
   (c) (5) Find $(R3)^{201}$.
   (d) (-)(5) Is $D_{10}$ Abelian? Give a short reason.

8. (20) Factor $2^3 \cdot 3 \cdot 5 \cdot 13$ into primes in $\mathbb{Z}[i]$. Prove that if $p = 4k + 3$ is a prime then it is a Gaussian prime.

9. (20) This problem concerns finite Abelian Groups.
   
   (a) (5) State the Fundamental Theorem for finite Abelian Groups, the result that describes all such groups.
   (b) (5) How many elements of $\mathbb{Z}_5 \times \mathbb{Z}_{25}$ have order 5?
   (c) (10) List the Abelian Groups with precisely 100 elements. (Your list should have every such group but each group only once.)

10. (-)(10) In $\mathbb{Z}[i]$ is $9 + 4i \in (2 + i)$? Give your reason!

11. (35) This problem concerns the finite field (you do not need to prove it is a field!)
    
    $F = \mathbb{Z}_7[x]/(x^3 + 5)$
    
    (a) (5) Describe the elements of $F$. How many are there?
    (b) (5) Find $(x^2 + 3)(x^2 + 4)$.
    (c) (5) Find the multiplicative inverse of $x$.
    (d) (10) Find all solutions, and show that they are all solutions, to the equation $y^6 = 1$ in $F$.
    (e) (10) (*) Find $(x + 1)^{3420003}$.

It’s not what you don’t know that hurts you.
It’s what you know that just ain’t so.
– Satchel Paige
The Dihedral Group $D_{10}$ is the group of symmetries of the regular 5-gon. We imagine the vertices of the regular 5-gon labelled $0, 1, 2, 3, 4$ in counterclockwise direction.

The symmetries come in three forms:

1. $R_i, i = 1, 2, 3, 4$. This is a rotation by $i$ notches in the counterclockwise direction.

2. $F_i, i = 0, 1, 2, 3, 4$. This is a flip around point $i$.

3. The identity $e$.

A Table for the Dihedral Group $D_{10}$.

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