SAMPLE ALGEBRA MIDTERM with SOLUTIONS

1. Let $H$ be a subgroup of $G$. Let $x, y \in G$ with $yx^{-1} \in H$. Prove that $Hx = Hy$.
   Solution: $xy^{-1} = h \in H$ so $x = hy$ and $y = h^{-1}x$. For any $h_1 \in H$, $h_1x = (h_1h)y \in Hy$. So $Hx \subseteq Hy$. Conversely for any $h_2 \in H$, $h_2y = h_2h^{-1}x$ and $Hy \subseteq Hx$ so $Hx = Hy$.

2. Let $\phi : G \to H$ be a homomorphism and let $W = \{h \in H : \phi(g) = h \text{ for some } g \in G\}$
   Show that $W$ is a subgroup of $H$.
   Solution:
   (a) As $\phi(e_G) = e_H$, $e_H \in W$.
   (b) Suppose $x, y \in W$. There exist $g_1, g_2 \in G$ with $\phi(g_1) = x, \phi(g_2) = y$. Then $\phi(g_1g_2) = xy$ so $xy \in W$.
   (c) Suppose $x \in W$. There exists $g \in G$ with $\phi(g) = x$. But then $\phi(g^{-1}) = x^{-1}$ so $x^{-1} \in W$.

3. Let $p, q$ be distinct primes. Let $G$ be an Abelian group. Let $x, y \in G$ with $o(x) = p$ and $o(y) = q$. Prove $o(xy) = pq$.
   Solution: $(xy)^{pq} = (x^p)^q(y^q)^p = e^q e^p = e$. Hence $o(xy) | pq$. As $p, q$, $o(xy)$ is one of $1, p, q, pq$. $(xy)^p = x^py^p = ey^p = y^p$. As $o(y) = q$ and $p$ can’t be a multiple of $q$ (they are distinct primes), $y^p \neq e$. Thus $(xy)^p \neq e$. Similarly $(xy)^q \neq e$. And $(xy)^1 \neq e$ as if it did, $y = x^{-1}$ and $y^p = e$ which it doesn’t. So $o(xy)$ is none of $1, p, q$ so it must be $pq$.

4. Give the elements and table of $(\mathbb{Z}_8^*, \cdot)$. Are $(\mathbb{Z}_4, +)$ and $(\mathbb{Z}_8^*, \cdot)$ isomorphic? If yes, give the isomorphism. If no, give a reason they are not isomorphic.
   Solution: Elements are $1, 3, 5, 7$ (numbers from $1$ to $7$ relatively prime to $8$) Table:
   
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No, not isomorphic. ($\mathbb{Z}_4, +$) has an element of order 4 (namely, 1) whereas ($\mathbb{Z}_8^*, \cdot$) does not.

5. Let $G$ be an Abelian group, $p$ a prime integer, and set

$$W = \{g \in G : g^{p^i} = e \text{ for some } i \geq 1\}$$

Let $x, y \in W$. Prove $xy \in W$.

Solution: There exist $i, j$ with $x^{p^i} = y^{p^j} = e$. Suppose $i \leq j$. Then $x^{p^j} = (x^{p^i})^{p^{j-i}} = e^{p^{j-i}} = e$ so $(xy)^{p^j} = x^{p^j}y^{p^j} = e \cdot e = e$ and $xy \in W$. The case $j \leq i$ is similar.

6. Give examples (no proofs needed) of:

(a) A group with precisely 127 elements
   Solution: ($\mathbb{Z}_{127}$, $+$)

(b) An infinite group which is not Abelian
   $GL_2(\mathbb{R})$, the group of nonsingular $2 \times 2$ matrices with real coefficients under multiplication.

(c) $H, G$ where $H$ is a subgroup of $G$ but $H$ is not a normal subgroup of $G$
   Solution: $G = S_3$, $H = \{e, (12) \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}\}$.

7. Let $G, g \in G$ and define $\phi : G \rightarrow G$ by $\phi(x) = gxg^{-1}$. Show that $\phi$ is a homomorphism. Show that $\phi$ has kernel $K = \{e\}$.
   Solution: $\phi(xy) = g^{-1}xyg = g^{-1}xgg^{-1}yg = \phi(x)\phi(y)$ so $\phi$ is a homomorphism. If $e = \phi(x) = gxg^{-1}$ then, multiplying by $g^{-1}$ on left and $g$ on right, $e = g^{-1}eg = g^{-1}gxg^{-1}g = x$. That is, the kernel $K = \{e\}$. 

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