COMPLEMENTARITY AND NONDEGENERACY IN SEMIDEFINITE PROGRAMMING

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Abstract. Primal and dual nondegeneracy conditions are defined for semidefinite programming. Given the existence of primal and dual solutions, it is shown that primal nondegeneracy implies a unique dual solution and that dual nondegeneracy implies a unique primal solution. The converses hold if strict complementarity is assumed. Primal and dual nondegeneracy assumptions do not imply strict complementarity, as they do in LP. The primal and dual nondegeneracy assumptions imply a range of possible ranks for primal and dual solutions X and Z. This is in contrast with LP where nondegeneracy assumptions exactly determine the number of variables which are zero. It is shown that primal and dual nondegeneracy and strict complementarity all hold generically. Numerical experiments suggest probability distributions for the ranks of X and Z which are consistent with the nondegeneracy conditions.

1. Duality and Complementarity

Let $\mathcal{S}^n$ denote the set of real symmetric $n \times n$ matrices. Denote the dimension of this space by

$$n^{\overline{2}} = n(n+1)/2. \tag{1}$$

The standard inner product on $\mathcal{S}^n$ is

$$A \bullet B = \mathbf{tr}\, AB = \sum_{i,j} a_{ij} b_{ij}.$$

By $X \succeq 0$, where $X \in \mathcal{S}^n$, we mean that X is positive semidefinite. The set $\mathcal{K} = \{X \in \mathcal{S}^n : X \succeq 0\}$ is called the positive semidefinite cone. The constraint $X \succeq 0$ is equivalent to a bound constraint on the least eigenvalue of X, which is not a differentiable function of X.

Consider the semidefinite programming (SDP)

$$\begin{array}{ll} \min & C \bullet X \\ \text{s.t.} & A_k \bullet X = b_k \quad k = 1, \ldots, m; \quad X \succeq 0. \end{array} \tag{2}$$

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Here C and A_k, $k = 1, \dots, m$, are all fixed matrices in $\mathcal{S}^n$, and the unknown variable X also lies in $\mathcal{S}^n$. The dual of SDP is

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & \sum_{k=1}^m y_k A_k + Z = C; \quad Z \succeq 0 \end{array} \tag{3}$$

where Z is a dual slack matrix variable, which also lies in $\mathcal{S}^n$. More generally, one may consider SDP in a space of block diagonal symmetric matrices, reducing to linear programming (LP) in the case that all block sizes are one. All definitions and results in this paper are easily extended to the block diagonal case.

For feasible X, y, Z the duality gap is $X \bullet Z = \mathbf{tr}\ XZ$, since

$$C \bullet X - b^T y = Z \bullet X + \sum_{k=1}^m y_k A_k \bullet X - b^T y = X \bullet Z \geq 0.$$

The following are assumed to hold throughout the paper.

Assumption 1. There exists a primal feasible point $X \succ 0$, and a dual feasible point (y, Z) with $Z \succ 0$.

Assumption 2. The matrices A_k, $k = 1, \dots, m$, are linearly independent, i.e. they span an m-dimensional linear space in $\mathcal{S}^n$.

Assumption 1 (a Slater condition) implies (see e.g. [NN94]) that the duality gap $X \bullet Z = 0$ for optimal solutions (X, y, Z). As is well known, this implies the complementary condition

$$XZ = 0. \tag{4}$$

To prove this, observe that $X \succeq 0$, $Z \succeq 0$ and $\mathbf{tr}\ XZ = 0$ imply that the matrix $X^{1/2} Z X^{1/2}$ is symmetric, positive semidefinite, and has zero trace. It follows that $X^{1/2} Z X^{1/2} = 0$, and therefore that $XZ = 0$.

The complementarity condition (4) implies that X and Z commute, so they share a common system of eigenvectors. Thus we have:

Lemma 1. *Let X and (y, Z) be respectively primal and dual feasible. Then they are optimal if and only if there exists $Q \in \mathbf{R}^{n \times n}$, with $Q^T Q = I$, such that*

$$X = Q\ \mathbf{Diag}(\lambda_1, \dots, \lambda_n)\ Q^T, \tag{5}$$

$$Z = Q\ \mathbf{Diag}(\omega_1, \dots, \omega_n)\ Q^T, \tag{6}$$

and

$$\lambda_i \omega_i = 0,\ \ i = 1, \cdots, n \tag{7}$$

all hold.

Equation (7) expresses complementarity in terms of the eigenvalues of X and Z. If X has rank r and Z has rank s, complementarity implies $r + s \leq n$.

Definition 1. Assume $XZ = 0$, with r and s respectively the ranks of X and Z. *Strict complementarity* holds if $r + s = n$, i.e. for each $i \in \{1, \ldots, n\}$, exactly one of the two conditions $\lambda_i = 0$ and $\omega_i = 0$ holds. We say that a semidefinite program satisfies strict complementarity if strict complementarity holds for every primal feasible X and dual feasible Z satisfying $XZ = 0$.

2. Nondegeneracy and Strict Complementarity

In this section we define nondegeneracy for SDP. To some extent our discussion is motivated by work on eigenvalue optimization by Overton and Womersley [OW95] and Shapiro and Fan [SF95]. Shapiro [Sha96] gives related results and extends them to nonlinear SDP's.

Consider the set

$$\mathcal{M}_r = \{X \in \mathcal{S}^n : \ \mathbf{rank}\,(X) = r\}.$$

Since the eigenvalues of a matrix X are continuous functions of X, it is clear that, for $r > 0$, the boundary of $\mathcal{M}_r$ is

$$\partial\mathcal{M}_r = \mathcal{M}_0 \cup \cdots \cup \mathcal{M}_{r-1}.$$

Let

$$\mathcal{M}_r^+ = \mathcal{K} \cap \mathcal{M}_r = \{X \in \mathcal{S}^n : \ X \succeq 0 \text{ and } \mathbf{rank}\,(X) = r\}.$$

Then the boundary of $\mathcal{K}$ is given by

$$\partial\mathcal{K} = \mathcal{M}_0^+ \cup \cdots \cup \mathcal{M}_{n-1}^+ \tag{8}$$

and the interior of $\mathcal{K}$ is

$$\text{Int } \mathcal{K} = \mathcal{M}_n^+.$$

Before going further, let us consider analogous definitions for the nonnegative orthant $\mathcal{J} = \{x \in \mathbf{R}^n \ : x \geq 0\}$. Consider the set

$$\mathcal{L}_r = \{x \ \in \mathbf{R}^n \ : \ x \text{ has exactly } r \text{ nonzero elements}\}.$$

For $r > 0$ the boundary of $\mathcal{L}_r$ is

$$\partial\mathcal{L}_r = \mathcal{L}_0 \cup \cdots \cup \mathcal{L}_{r-1}.$$

Let

$$\mathcal{L}_r^+ = \mathcal{J} \cap \mathcal{L}_r.$$

The boundary of $\mathcal{J}$ is

$$\partial\mathcal{J} = \mathcal{L}_0^+ \cup \cdots \cup \mathcal{L}_{n-1}^+ \tag{9}$$

and the interior of $\mathcal{J}$ is

$$\text{Int } \mathcal{J} = \mathcal{L}_n^+.$$

However, the decompositions (8) and (9) have very different characters. The set $\mathcal{L}_r^+$ is not connected, except in the cases $r = 0$ and $r = n$. For example, for $n = 2$, the set $\mathcal{L}_1^+$ consists of the two positive coordinate axes (excluding the origin). By contrast, the set $\mathcal{M}_r^+$ is a path-connected smooth submanifold of $\mathcal{S}^n$ for all r, $0 \leq r \leq n$. For example, in the case $n = 2$, we have

$$\mathcal{M}_1^+ = \left\{ \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix} : \; \alpha \geq 0, \; \beta \geq 0, \alpha + \beta > 0, \; \gamma = \pm\sqrt{\alpha\beta} \right\},$$

a connected, smooth submanifold of $\mathcal{S}^2$.

Let X be primal feasible with $\mathbf{rank}\,(X) = r$ and

$$X = Q \; \mathbf{Diag}(\lambda_1, \ldots, \lambda_r, 0, \cdots, 0) \; Q^T \tag{10}$$

where $Q^T Q = I$. The tangent space to $\mathcal{M}_r$ at X is [Arn71]

$$\mathcal{T}_X = \left\{ Q \begin{bmatrix} U & V \\ V^T & 0 \end{bmatrix} Q^T \; : \; U \in \mathcal{S}^r, V \in \mathbf{R}^{r \times n-r} \right\}.$$

Recalling the notation (1), $\dim \mathcal{T}_X = r^{\overline{2}} + r(n-r) = n^{\overline{2}} - (n-r)^{\overline{2}}$. For $\Delta X \in \mathcal{T}_X$ we have

$$Q^T(X + \epsilon \Delta X)Q = \begin{bmatrix} \mathbf{Diag}(\lambda_1, \ldots, \lambda_r) + \epsilon U & \epsilon V \\ \epsilon V^T & 0 \end{bmatrix}.$$

Thus $X \pm \epsilon\Delta X$ is *not* contained in $\mathcal{K}$, for $\epsilon > 0$, unless $V = 0$.

Definition 2. X is *primal nondegenerate* if it is primal feasible and

$$\mathcal{T}_X + \mathcal{N} = \mathcal{S}^n, \tag{11}$$

where

$$\mathcal{N} = \{ Y \in \mathcal{S}^n : \; A_k \bullet Y = 0 \text{ for all } k \}. \tag{12}$$

We say that a semidefinite program satisfies primal nondegeneracy if every primal feasible X is primal nondegenerate.

Theorem 1. *Let X be primal feasible with* $\mathbf{rank}\,(X) = r$. *A necessary condition for X to be primal nondegenerate is that*

$$(n-r)^{\overline{2}} \leq n^{\overline{2}} - m. \tag{13}$$

Furthermore, let $Q_1 \in \mathbf{R}^{n \times r}$ and $Q_2 \in \mathbf{R}^{n \times (n-r)}$ respectively denote the first r columns and the last $n - r$ columns of Q given by (10). Then X is primal nondegenerate if and only if the matrices

$$B_k = \begin{bmatrix} Q_1^T A_k Q_1 & Q_1^T A_k Q_2 \\ Q_2^T A_k Q_1 & 0 \end{bmatrix}, \; k = 1, \ldots, m \tag{14}$$

are linearly independent in $\mathcal{S}^n$.

Proof. Inequality (13) follows directly from Definition 2, since $\dim \mathcal{T}_X = n^{\overline{2}} - (n-r)^{\overline{2}}$ and $\dim \mathcal{N} = n^{\overline{2}} - m$. Equation (11) is equivalent to

$$\mathcal{T}_X^\perp \cap \mathcal{N}^\perp = \{0\} \tag{15}$$

where $\mathcal{T}_X^\perp$ and $\mathcal{N}^\perp$ are respectively the orthogonal complements of $\mathcal{T}_X$ and $\mathcal{N}$, namely

$$\mathcal{T}_X^\perp = \left\{ Q \begin{bmatrix} 0 & 0 \\ 0 & W \end{bmatrix} Q^T \ : \ W \in \mathcal{S}^{n-r} \right\}$$

and

$$\mathcal{N}^\perp = \mathrm{Span}\{A_k\}.$$

If the B_k are linearly dependent, there exist θ_k not all zero such that $\sum \theta_k B_k = 0$. This contradicts (15), since then $\sum \theta_k A_k \in \mathcal{T}_X^\perp$. Conversely, if the B_k are linearly independent, (15) holds. □

Note that Theorem 1 holds for any Q satisfying (10).

Theorem 2. *Let X be primal nondegenerate and optimal. Then there exists a unique optimal dual solution (y, Z).*

Proof. By Assumption 2, a dual optimal solution (y, Z) exists, so that complementarity holds. As above, let Q_1 and Q_2 respectively denote the first r columns and the last $n - r$ columns of Q given in (10). Any $\widetilde{Z}$ satisfying the complementarity condition $X\widetilde{Z} = 0$ must be of the form

$$\widetilde{Z} = Q_2 W Q_2^T$$

for some $W \in \mathcal{S}^{n-r}$, so the feasibility condition (3) requires the existence of $\tilde{y} \in \mathbf{R}^m$ and $W \in \mathcal{S}^{n-r}$ such that

$$Q_2 W Q_2^T + \sum_{k=1}^{m} \tilde{y}_k A_k = C.$$

Theorem 1 guarantees that any solution of this linear system is unique. □

Note that if we assume Q satisfies (6) as well as (10) we find that $W = \mathbf{Diag}(\omega_{r+1}, \ldots, \omega_n)$.

Now we turn to dual nondegeneracy. Let (y, Z) be dual feasible with $\mathbf{rank}\,(Z) = s$ and

$$Z = Q\ \mathbf{Diag}(0, \ldots, 0, \omega_{n-s+1}, \ldots, \omega_n)\ Q^T \tag{16}$$

with $Q^T Q = I$. The tangent space to $\mathcal{M}_s$ at Z is

$$\mathcal{T}_Z = \left\{ Q \begin{bmatrix} 0 & V \\ V^T & W \end{bmatrix} Q^T \ : \ V \in \mathbf{R}^{(n-s)\times s},\ W \in \mathcal{S}^s \right\}. \tag{17}$$

We have $\dim(\mathcal{T}_Z) = \overline{s^2} + s(n-s) = \overline{n^2} - \overline{(n-s)^2}$.

Definition 3. The point (y, Z) is *dual nondegenerate* if it is dual feasible and Z satisfies

$$\mathcal{T}_Z + \mathrm{Span}\{A_k\} = \mathcal{S}^n. \tag{18}$$

We say that a semidefinite program satisfies dual nondegeneracy if every dual feasible (y, Z) is dual nondegenerate.

Theorem 3. *Let (y, Z) be dual feasible with* $\mathbf{rank}\,(Z) = s$. *A necessary condition for (y, Z) to be dual nondegenerate is that*

$$\overline{(n-s)}^{2} \le m. \tag{19}$$

Furthermore, let $\widetilde{Q}_1 \in \mathbf{R}^{n\times(n-s)}$ and $\widetilde{Q}_2 \in \mathbf{R}^{n\times s}$ respectively denote the first $n-s$ and the last s columns of Q given by (16). Then (y, Z) is dual nondegenerate if and only if the matrices

$$\widetilde{B}_k = \left[\widetilde{Q}_1^T A_k \widetilde{Q}_1\right],\ \ k = 1, \ldots, m \tag{20}$$

span $\mathcal{S}^{n-s}$.

Proof. It is an immediate consequence of the definition. □

Note that Theorem 3 holds for any Q satisfying (16).

Theorem 4. *Let (y, Z) be dual nondegenerate and optimal. Then there exists a unique optimal primal solution X.*

Proof. By Assumption 2, a primal optimal solution X exists. As above let $\widetilde{Q}_1$ and $\widetilde{Q}_2$ respectively denote the first $n-s$ columns and the last s columns of Q given by (16). Any $\widetilde{X}$ satisfying the complementarity condition $\widetilde{X}Z = 0$ must be of the form

$$\widetilde{X} = \widetilde{Q}_1 U \widetilde{Q}_1^T$$

for some $U \in \mathcal{S}^{n-s}$. Thus the feasibility condition (2) reduces to

$$\left(\widetilde{Q}_1^T A_k \widetilde{Q}_1\right) \bullet U = b_k,\ \ k = 1, \ldots, m \tag{21}$$

Theorem 3 guarantees that any solution of this linear system is unique. □

Note that if we assume Q satisfies (5) as well as (16) we find that $U = \mathbf{Diag}(\lambda_1, \ldots, \lambda_{n-s})$.

Note also the distinction between the partitionings of Q used in Theorems 1 and 3. The former uses $Q = [Q_1\ Q_2]$ where Q_1 has r columns and the latter uses $Q = [\widetilde{Q}_1\ \widetilde{Q}_2]$ where $\widetilde{Q}_1$ has $n-s$ columns. These partitionings are the same if and only if $r + s = n$, i.e. strict complementarity holds.

It is instructive to compare our SDP nondegeneracy definitions with those of the LP

$$\min\ c^T x \ \text{ subject to } Ax = b,\ x \ge 0,$$

where $A \in \mathbf{R}^{m\times n}$. Suppose that r is the number of nonzero primal solution variables x_j, with $x_{r+1} = \cdots = x_n = 0$, and s is the number of corresponding nonzero dual slacks z_j, with $z_1 = \cdots = z_{n-s} = 0$. By complementarity, $r + s \le n$. Consider the partitionings

$$A = [A_1\ A_2] \text{ and } A = [\widetilde{A}_1\ \widetilde{A}_2]$$

where A_1 has r columns and $\widetilde{A}_1$ has $n-s$ columns. These partitionings are identical if strict complementarity holds. LP primal nondegeneracy states that the m rows of A_1 must be a linearly independent set in $\mathbf{R}^r$, which requires $r \ge m$. LP dual nondegeneracy states that the m rows of $\widetilde{A}_1$ should span $\mathbf{R}^{n-s}$, which requires $s \geq n - m$. Combined with the complementarity condition $r + s \leq n$, these conditions imply $r = m$ and $s = n - m$. Thus in LP, primal and dual nondegeneracy imply strict complementarity. This is *not* the case for SDP.

Example. Let $n = 3, m = 3$, with $b = [1\ 0\ 0]^T$,

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which has the solution

$$X = \mathbf{Diag}(1,0,0),\ y = [0\ 0\ 0],\ Z = \mathbf{Diag}(0,0,1).$$

That this solution is valid is easily verified by checking the optimality conditions (2), (3), (4). We have $Q = I$, with the eigenvalues λ_i and ω_i equal to the diagonal entries of X and Z respectively. Note that $r = 1$ and $s = 1$, so strict complementarity does *not* hold. Let us check the primal nondegeneracy condition. We have

$$Q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ Q_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so the matrices B_k, $k = 1, 2, 3$, defined by (14), are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since these are linearly independent, the primal nondegeneracy condition holds, and the dual solution must be unique. Now let us check the dual nondegeneracy condition. We have

$$\widetilde{Q}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and the matrices $\widetilde{B}_k$, $k = 1, 2, 3$, defined by (20), are given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since these span $\mathcal{S}^2$, the dual nondegeneracy condition holds, and the primal solution must be unique. Note especially that, in this example, strict complementarity fails to hold even in the presence of primal and dual nondegeneracy.

Theorems 2 and 4 show that primal and dual nondegeneracy respectively imply dual and primal unique solutions. The converses are true assuming strict complementarity:

Theorem 5. *Suppose that X and (y, Z) are respectively primal and dual optimal solutions satisfying strict complementarity. Then if the primal solution X is unique, the dual nondegeneracy condition must hold, and if the dual solution (y, Z) is unique, the primal nondegeneracy condition must hold.*

Proof. Let Q satisfy conditions (10) and (16), as in Lemma 1. Strict complementarity states that $r + s = n$, so the partitionings of Q used in Theorems 1 and 3 are the same. Thus

$$X = Q_1 \ \mathbf{Diag}(\lambda_1, \ldots, \lambda_r) \ Q_1^T, \quad Z = Q_2 \ \mathbf{Diag}(\omega_{r+1}, \ldots, \omega_n) \ Q_2^T.$$

Suppose first that the dual nondegeneracy assumption (18) fails to hold. We shall show that in this case X cannot be a unique primal solution. Since Z is an optimal dual solution, complementarity states that any optimal primal solution $\widetilde{X}$ must satisfy

$$\widetilde{X} = Q_1 U Q_1^T$$

for some $U \in \mathcal{S}^r$, and so the primal feasibility condition (2) reduces to

$$\left(Q_1^T A_k Q_1\right) \bullet U = b_k, \ \ k = 1, \ldots, m.$$

Because the dual nondegeneracy assumption does not hold, the solution set of this equation is *not* unique, but holds on an affine subset of $\mathcal{S}^r$, say $\mathcal{U}$, with positive dimension. The condition that $\widetilde{X} \succeq 0$ holds if and only if $U \succeq 0$. But the particular choice $U = \mathbf{Diag}(\lambda_1, \ldots, \lambda_r)$ lies in $\mathcal{U}$ and is positive definite, so there is an open set in $\mathcal{U}$ for which the same is true. Every such U defines an $\widetilde{X}$ which satisfies the optimality conditions.

Now suppose that the primal nondegeneracy assumption (11) fails to hold. We shall show that in this case (y, Z) cannot be a unique dual solution. Complementarity states that any solution $\widetilde{Z}$ must satisfy

$$\widetilde{Z} = Q_2 W Q_2^T$$

for some $W \in \mathcal{S}^s$, and so the dual feasibility condition (3) reduces to the solvability of

$$Q_2 W Q_2^T + \sum_{k=1}^{m} \tilde{y}_k A_k = C$$

for some $\tilde{y} \in \mathbf{R}^m$ and $W \in \mathcal{S}^s$. Because the primal nondegeneracy assumption does not hold, the solution set of this equation is *not* unique, but holds on an affine subset of $\mathcal{S}^s \times \mathbf{R}^m$, say $\mathcal{W}$, with positive dimension. The condition $\widetilde{Z} \succeq 0$ in (3) holds if and only if $W \succeq 0$. But the particular choice ($\tilde{y} = y, W = \mathbf{Diag}(\omega_{r+1}, \ldots, \omega_n)$) lies in $\mathcal{W}$ with W positive definite, so there is an open set in $\mathcal{W}$ for which the same is true. Every such W defines a $\widetilde{Z}$ which satisfies the optimality conditions. $\square$

If the assumption of strict complementarity is not made, it is possible that the primal solution is unique even if the dual nondegeneracy assumption fails. Consider Example 1, changing it so that

$$A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \widetilde{B}_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and therefore the dual nondegeneracy assumption does not hold. It follows that U is not uniquely defined by (21): we can take

$$U = \begin{bmatrix} 1 & \theta \\ \theta & 0 \end{bmatrix}$$

for any $\theta \in \mathbf{R}$. However, only $\theta = 0$ gives $U \succeq 0$ and therefore $\widetilde{X} \succeq 0$.

It is convenient to introduce some further notation at this point. Let

$$\sqrt[\overline{2}]{k} = \lfloor h \rfloor \quad \text{where } h \text{ is the positive real root of } h^{\overline{2}} = k.$$

We then have:

Theorem 6. *Suppose that X and (y, Z) are respectively primal and dual nondegenerate and optimal, with* $\mathbf{rank}\,(X) = r$ *and* $\mathbf{rank}\,(Z) = s$. *Then*

$$n - \sqrt[\overline{2}]{n^{\overline{2}} - m} \;\le\; r \;\le\; \sqrt[\overline{2}]{m} \tag{22}$$

and

$$n - \sqrt[\overline{2}]{m} \;\le\; s \;\le\; \sqrt[\overline{2}]{n^{\overline{2}} - m}. \tag{23}$$

Proof. The lower bounds in (22), (23) are the necessary conditions (13) and (19) given by Theorems 1 and 3. The upper bounds follow from the complementarity condition $r + s \le n$. ☐

The ranges of possible values for the ranks of solutions X and Z stand in contrast with LP, where nondegeneracy assumptions give precise formulas for the number of nonzero primal and dual variables. In fact, (22), (23) reduce to equalities only in the cases $m = 0$ ($r = 0$, $s = n$) and $m = n^{\overline{2}}$ ($r = n$, $s = 0$).

Pataki [Pat95] has shown that there always exist optimal solutions X and Z satisfying the *upper* bounds in (22) and (23). Nondegeneracy assumptions are not required for these results. However, without nondegeneracy assumptions the upper bounds need not hold for all solutions, and the lower bounds may not hold for any solution.

We now compare our nondegeneracy conditions with that given by Anderson and Nash [AN87, p.21] in the context of infinite-dimensional LP over general cones. Let $\mathcal{B}_X$ be the linear span of the face of $\mathcal{K}$ generated by X (the face of $\mathcal{K}$ containing X and having minimal dimension). The Anderson-Nash nondegeneracy condition applied to SDP is

$$\mathcal{B}_X + \mathcal{N} = \mathcal{S}^n. \tag{24}$$

It is well known, e.g. [Tau67, p.182], that

$$\mathcal{B}_X = \left\{ Q \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} Q^T \;:\; U \in \mathcal{S}^r \right\}, \tag{25}$$

where Q satisfies (10). To prove this, note that for ΔX in (25),

$$Q^T(X + \epsilon \Delta X)Q = \begin{bmatrix} \mathbf{Diag}(\lambda_1, \dots, \lambda_r) + \epsilon U & 0 \\ 0 & 0 \end{bmatrix}$$

so that $X \pm \epsilon \Delta X \in \mathcal{M}_r^+ \subset \mathcal{K}$ for sufficiently small ϵ. This is not true for $\Delta X \notin \mathcal{B}_X$. We have $\mathcal{B}_X \subset \mathcal{T}_X$, with dim $\mathcal{B}_X = \overline{r^2}$. Likewise

$$\mathcal{B}_Z = \left\{ Q \begin{bmatrix} 0 & 0 \\ 0 & W \end{bmatrix} Q^T \; : \; W \in \mathcal{S}^s \right\}$$

where Q satisfies (16), and so the Anderson-Nash nondegeneracy condition applied to the dual is

$$\mathcal{B}_Z + \mathrm{Span}\{A_k\} = \mathcal{S}^n. \tag{26}$$

Assumptions (24) and (26) imply that

$$\overline{r^2} \geq m \quad \text{and} \quad \overline{s^2} \geq \overline{n^2} - m$$

must both hold. However, these inequalities can never hold simultaneously, except in the trivial cases $m = 0$ and $m = \overline{n^2}$, because $r + s \leq n$. The relationship between the Anderson-Nash conditions and ours is clarified by noting that (13) and (19) can be written

$$\overline{r^2} + r(n-r) \geq m \quad \text{and} \quad \overline{s^2} + s(n-s) \geq \overline{n^2} - m.$$

Anderson and Nash define X to be basic if

$$\mathcal{B}_X \cap \mathcal{N} = \{0\}. \tag{27}$$

They show that a point is basic if and only if it is an extreme point of the feasible set, and, that if an optimal solution exists, there is a basic optimal solution. This provides another way to recover Pataki's results since (27) implies

$$\dim \mathcal{B}_X \leq \overline{n^2} - \dim \mathcal{N}$$

i.e.

$$\overline{r^2} \leq m.$$

We also have:

Theorem 7. *Suppose that X and (y, Z) are respectively primal and dual optimal solutions satisfying strict complementarity. Then X is basic if and only if the dual nondegeneracy condition holds.*

Proof. Since $r + s = n$, we have $\mathcal{B}_X = \mathcal{T}_Z^\perp$ (see (25) and (17)). Thus the condition that X be primal basic, namely (27), is equivalent to the condition that (y, Z) be dual nondegenerate, namely (18). □

3. Genericity and Transversality

Primal nondegeneracy, dual nondegeneracy and strict complementarity are all generic properties of semidefinite programs, in a sense that we shall make precise. We begin by noting that, assuming strict complementarity is a generic property, it is intuitively obvious from Theorem 5 that the primal and dual nondegeneracy conditions hold generically for optimal points. This is because if, for example, the dual nondegeneracy condition fails to hold at an optimal solution, the primal solution is not unique, so the matrix C is orthogonal to a face of the primal feasible region. This is an obviously non-generic property. In fact, we show below that primal and dual nondegeneracy hold generically without any reference to strict complementarity.

The main results of this section are summarized in the next two theorems.

Theorem 8. *Primal nondegeneracy is a generic property of semidefinite programs. Similarly, dual nondegeneracy is generic.*

Theorem 9. *Strict complementarity is a generic property of semidefinite programs.*

In the remainder of this section, we precisely define the notion of genericity and then prove Theorems 8 and 9. The proofs use the notion of transversality from differential topology. A result similar to Theorem 8 is stated in [Sha96]. Theorem 9 does not seem to have appeared elsewhere. It is more appropriate for this discussion to write a semidefinite program in a self-dual form [NN94]. Let $\mathcal{L}$ denote the subspace of $\mathcal{S}^n$ spanned by $A_1, \ldots, A_m$. Then the primal problem (2) can be written as

$$\begin{array}{ll} \min & C \bullet X \\ \text{s.t.} & X \in D + \mathcal{L}^\perp, \quad X \succeq 0, \end{array} \tag{28}$$

while the dual problem (3) can be written as

$$\begin{array}{ll} \min & D \bullet Z \\ \text{s.t.} & Z \in C + \mathcal{L}, \quad Z \succeq 0. \end{array} \tag{29}$$

Thus, for a fixed n and m, a semidefinite program is completely determined by C, D, and $\mathcal{L}$ with $\dim \mathcal{L} = m$.

In what follows we must be careful to distinguish between a subspace $\mathcal{L}$ and the choice of basis defining it, which is not unique. Let $\mathbf{G}_{\overline{n^2},m}$ denote the manifold of all linear subspaces of dimension m in $\mathcal{S}^n$ (the Grassman manifold), and let $\mathbf{V}_{\overline{n^2},m}$ denote the manifold of orthonormal m-frames in $\mathcal{S}^n$ (the Stiefel manifold), where an orthonormal m-frame is a set of m pairwise orthogonal elements of $\mathcal{S}^n$ of unit length. Also let

$$\pi_1 : \mathbf{V}_{\overline{n^2},m} \longrightarrow \mathbf{G}_{\overline{n^2},m} \tag{30}$$

denote the natural map which associates with an orthonormal m-frame the subspace it spans. Both $\mathbf{G}_{\overline{n^2},m}$ and $\mathbf{V}_{\overline{n^2},m}$ are smooth manifolds and π_1 is a smooth map [Boo75]. It is clear from (28) (or (29)) that semidefinite programs are parametrized by triple

$$(C, D, \mathcal{L}) \in \mathcal{S}^n \times \mathcal{S}^n \times \mathbf{G}_{\overline{n^2},m}.$$

We need the following basic definitions.

Definition 4. A subset E of $\mathbf{R}^k$ is said to have measure 0 if for every $\epsilon > 0$ there exists a countable collection of k-dimensional cubes (or balls) $\mathcal{C}_1, \mathcal{C}_2, \ldots$ whose union contains E and whose total volume is less than ϵ.

Definition 5. A subset E of a k-dimensional manifold M is said to have measure 0 if $\phi(E \cap U)$ has measure 0 in $\mathbf{R}^k$ for every chart (U, ϕ) of M (see [GP74, p. 39]).

Remark 1. The map π_1 in (30) is *locally trivial* in the following sense: every point $x \in \mathbf{G}_{\overline{n^2},m}$ has an open neighborhood U whose inverse image, $\pi_1^{-1}(U) \subset \mathbf{V}_{\overline{n^2},m}$, is diffeomorphic to the product $U \times \mathcal{O}^m$ (see [Hu66]). Here $\mathcal{O}^m$ denotes the Lie group of orthogonal $m \times m$ matrices. One can then show that a subset $E \subset \mathbf{G}_{\overline{n^2},m}$ has measure 0 in $\mathbf{G}_{\overline{n^2},m}$ if and only if its inverse image $\pi_1^{-1}(E)$ has measure 0 in $\mathbf{V}_{\overline{n^2},m}$.

We are now ready to give a precise definition of genericity.

Definition 6. We say that a property $\mathcal{P}$ of semidefinite programs is generic if it holds for almost all triples $(C, D, \mathcal{L})$, that is the set of triples $(C, D, \mathcal{L})$ for which $\mathcal{P}$ fails to hold has measure 0 in $\mathcal{S}^n \times \mathcal{S}^n \times \mathbf{G}_{\overline{n^2},m}$.

Definition 7. Let $f : M \longrightarrow N$ be a smooth map between smooth manifolds M and N and let $K \subset N$ be a submanifold. We say that f is transversal to K at a point $x \in M$ if either $f(x) \notin K$ or else

$$df(x)(\mathcal{T}_x M) + \mathcal{T}_{f(x)} K = \mathcal{T}_{f(x)} N. \tag{31}$$

Here $\mathcal{T}_x M$ denotes the tangent space to M at x and $df(x)$ denotes the differential of f at x. We say that f is transversal to K if the transversality condition (31) is satisfied for every $x \in M$.

The proof of the following Theorem can be found in [GP74, p. 68].

Theorem 10. *Let $F : M \times T \longrightarrow N$ be a smooth map between smooth manifolds and let K be a submanifold of N. For $t \in T$ define a smooth map f_t from M to N by $f_t(x) = F(x, t)$. Suppose that F is transversal to K. Then f_t is transversal to K for almost all $t \in T$.*

Proof of Theorem 8: Let $\widetilde{E}$ denote the set of $(C, D, \mathcal{L}) \in \mathcal{S}^n \times \mathcal{S}^n \times \mathbf{G}_{\overline{n^2},m}$ for which dual nondegeneracy fails. We need to show that $\widetilde{E}$ has measure 0. In light of Remark 1 it is sufficient to prove that the set E of all

$$(C, D, A_1, \ldots, A_m) \in \mathcal{S}^n \times \mathcal{S}^n \times \mathbf{V}_{\overline{n^2},m}$$

for which dual nondegeneracy fails has measure 0. Consider the map

$$F : \mathbf{R}^m \times \mathcal{S}^n \times \mathcal{S}^n \times \mathbf{V}_{\overline{n^2},m} \longrightarrow \mathcal{S}^n$$

given by

$$F(y, C, D, A_1, \ldots, A_m) = C + \sum_{i=1}^m y_i A_i.$$

The differential of this map has maximal rank at every point, i.e. F is a submersion [GP74, p. 20]. It follows that F is transversal to any submanifold of $\mathcal{S}^n$. In particular it is transversal to each of the submanifolds $\mathcal{M}_s$, $s = 0, \ldots, n$. For a fixed $t = (C, D, A_1, \ldots, A_m)$, let

$$f_t : \mathbf{R}^m \longrightarrow \mathcal{S}^n$$

be given by

$$f_t(y) = F(y, C, D, A_1, \ldots, A_m).$$

Clearly, we have $df_t(y)(\mathbf{R}^m) = \mathrm{Span}\{A_k\}$. Thus the transversality condition (31) for f_t at y is precisely equivalent to the dual nondegeneracy condition (18) at the point $(y, f_t(y))$. Applying Theorem 10 to the map F we conclude that the set

$$E_s = \{t = (C, D, A_1, \ldots, A_m) \mid f_t \text{ is not transversal to } \mathcal{M}_s\}$$

has measure 0. Thus the set $E_0 \cup \cdots \cup E_n \subset \mathcal{S}^n \times \mathcal{S}^n \times \mathbf{V}_{\overline{n^2}, m}$ has measure 0 and it suffices to show that this set contains E. But if the semidefinite program determined by $t = (C, D, A_1, \ldots, A_m)$ does not satisfy dual nondegeneracy, then there exists $y \in \mathbf{R}^m$ and an integer s with $0 \leq s \leq n$ such that $Z = f_t(y) \in \mathcal{M}_s$, $Z \succeq 0$, but f is not transversal to $\mathcal{M}_s$ at y. Hence $(C, D, A_1, \ldots, A_m) \in E_s$. The proof that primal nondegeneracy is generic is similar. $\square$

In order to prove Theorem 9 we need the following key lemma.

Lemma 2. *Let r, s satisfy $0 \leq r, s \leq n$ and $r + s \leq n$. The subset $\mathcal{W}_{r,s}$ of $\mathcal{S}^n \times \mathcal{S}^n$ given by*

$$\mathcal{W}_{r,s} = \{(X, Z) \mid \mathbf{rank}\, X = r, \mathbf{rank}\, Z = s, \text{ and } XZ = 0\}$$

is a submanifold of dimension

$$d = \overline{n^2} - \overline{(n - (r+s))^2}.$$

Proof. It is sufficient to prove that for every $(X_0, Z_0) \in \mathcal{W}_{r,s}$ there exist open sets $U \subset \mathcal{S}^n \times \mathcal{S}^n$ containing (X_0, Z_0) and $V \subset \mathbf{R}^d$ and a differentiable map $h : V \longrightarrow U$ that is a homeomorphism from V onto $U \cap \mathcal{W}_{r,s}$.

Since X_0 is of rank r, we can assume, by permuting the rows and columns of both X_0 and Z_0, that X_0 is of the form

$$X_0 = \begin{bmatrix} A_0 & B_0^T \\ B_0 & B_0 A_0^{-1} B_0^T \end{bmatrix}$$

with $A_0 \in \mathcal{S}^r$ nonsingular, and $X_0 Z_0$ is still equal to 0. Now write Z_0 as

$$Z_0 = \begin{bmatrix} E_0 & F_0^T \\ F_0 & G_0 \end{bmatrix}$$

where $E_0 \in \mathcal{S}^r$. From $X_0 Z_0 = 0$ we have that

$$\begin{cases} F_0 &= G_0 H \\ E_0 &= H^T F_0 \end{cases}$$

where $H = -B_0 A_0^{-1}$. Therefore,

$$Z_0 = \begin{bmatrix} H^T G_0 H & H^T G_0 \\ G_0 H & G_0 \end{bmatrix}$$

and we conclude that $G_0 \in \mathcal{S}^{n-r}$ satisfies

$$\mathbf{rank}\, G_0 = \mathbf{rank}\, Z_0 = s.$$

Again, by permuting rows and columns of Z_0 and X_0, we may assume that G_0 is of the form

$$G_0 = \begin{bmatrix} K_0^T L_0^{-1} K_0 & K_0^T \\ K_0 & L_0 \end{bmatrix}$$

where $L_0 \in \mathcal{S}^s$ is nonsingular. Note that the 2×2 block structure of X_0 and Z_0 are not modified by these permutations. Furthermore, K_0 is empty if $s = n - r$. Now there exists an $\epsilon > 0$ such that

$$A \in \mathcal{S}^r,\ \|A - A_0\| < \epsilon \Longrightarrow A \text{ is nonsingular}$$

and

$$L \in \mathcal{S}^s,\ \|L - L_0\| < \epsilon \Longrightarrow L \text{ is nonsingular.}$$

Let $U \subset \mathcal{S}^n \times \mathcal{S}^n$ be the open set consisting of those pairs (X, Z) with

$$X = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}, Z = \begin{bmatrix} E & F^T \\ F & G \end{bmatrix}, \text{ with } G = \begin{bmatrix} M & K^T \\ K & L \end{bmatrix}$$

for which $\|A - A_0\| < \epsilon$ and $\|L - L_0\| < \epsilon$. Let

$$V \subset \mathcal{S}^r \times \mathbf{R}^{r\times(n-r)} \times \mathcal{S}^s \times \mathbf{R}^{s\times(n-r-s)} \cong \mathbf{R}^d$$

consist of all (A, B, L, K) such that $\|A - A_0\| < \epsilon$ and $\|L - L_0\| < \epsilon$. The map

$$h(A, B, L, K) = \left(\begin{bmatrix} A & B^T \\ B & BA^{-1}B^T \end{bmatrix}, \begin{bmatrix} A^{-1}B^T G B A^{-1} & -A^{-1}B^T G \\ -GBA^{-1} & G \end{bmatrix} \right)$$

where

$$G = \begin{bmatrix} K^T L^{-1} K & K^T \\ K & L \end{bmatrix}$$

meets all the requirements. □

Proof of Theorem 9: Let

$$T = \mathcal{S}^n \times \mathcal{S}^n \times \mathbf{G}_{\overline{n^2}, m},$$

and for $W \in \mathcal{S}^n$, $\mathcal{L} \in \mathbf{G}_{\overline{n^2}, m}$, let $W_{\mathcal{L}}$ and $W_{\mathcal{L}^\perp}$ denote the orthogonal projections of W onto $\mathcal{L}$ and $\mathcal{L}^\perp$ respectively. Now consider the map

$$F : \mathcal{S}^n \times T \longrightarrow \mathcal{S}^n \times \mathcal{S}^n$$

defined by

$$F(W, t) = (D + W_{\mathcal{L}^\perp}, C + W_{\mathcal{L}}), \quad \text{with } t = (C, D, \mathcal{L}).$$

Clearly the differential of F has maximal rank at every point, and so F is transversal to all the submanifolds $\mathcal{W}_{r,s}$ of $\mathcal{S}^n \times \mathcal{S}^n$. Applying Theorem 10, we conclude that for all r, s, with $r + s \leq n$, the maps $f_t(W) = F(W, t)$ are transversal to $\mathcal{W}_{r,s}$ for almost all $t \in T$.

Now let $r + s < n$ and suppose that f_t is transversal to $\mathcal{W}_{r,s}$. Then a simple dimension count, using (31) and Lemma 2, shows that $f_t(W) \notin \mathcal{W}_{r,s}$ for all $W \in \mathcal{S}^n$. We have proved that for almost all t the image of f_t does not intersect $\mathcal{W}_{r,s}$ whenever $r + s < n$. Thus the set E of all t such that the image of f_t intersects $\mathcal{W}_{r,s}$ for some r, s with $r + s < n$ has measure 0 in T. The result will follow if we show that the set $\widetilde{E}$ of all $(C, D, \mathcal{L})$ for which the corresponding semidefinite program does not satisfy strict complementarity is contained in E.

Let $(C, D, \mathcal{L})$ determine a semidefinite program that does not satisfy strict complementarity. Thus there exists a primal feasible X and a dual feasible Z such that $(X, Z) \in \mathcal{W}_{r,s}$ for some r, s with $r + s < n$. Let us write

$$X = D + L_1,\ Z = C + L_2, \quad \text{with } L_1 \in \mathcal{L}^\perp,\ L_2 \in \mathcal{L}$$

and let $W = (X - D) + (Z - C) \in \mathcal{S}^n$. Clearly, with $t = (C, D, \mathcal{L})$, we have $f_t(W) = (X, Z)$ lies in $\mathcal{W}_{r,s}$, so that $t \in E$. □

4. Rank Distributions

Now consider SDP's whose data are distributed according to a given probability distribution, e.g. uniformly in $[-1, 1]$, discarding those for which the Assumptions do not hold. We may consider the probability that solutions X and Z respectively have given ranks r and s. It follows from Theorem 8 that the probability that r and s satisfy the bounds in (22), (23) is one. Furthermore, it follows from Theorem 9 that the probability that $r + s = n$ is one. Consequently it is sufficient to consider r only. A natural question is: what is the probability distribution describing the values that r takes in the range (22)? We shall now show some experimental results addressing this question. This is a promising area for further theoretical investigation.

In this experiment, we set $n = 20$. The self-dual property of semidefinite programming implies that it is sufficient to consider one half of the possible range of values for m. We chose to let m range from 10 to 100 in increments of 10. For each value of m we considered a thousand randomly generated problems. Each problem was determined by the following data:

$$A_1, \ldots, A_m, C \in \mathcal{S}^n, \quad b \in \mathbf{R}^m.$$

The matrices $A_k, k = 1, \ldots, m$ were taken to be symmetric with entries uniformly distributed in the interval $[-1, 1]$. The vector b and the matrix C were chosen to ensure that Assumption 1 was satisfied. More precisely, random positive definite symmetric matrices $\widetilde{X}$ and $\widetilde{Z}$ and a random vector $\tilde{y} \in \mathbf{R}^m$ were generated, and b was defined by $b_k = A_k \bullet \widetilde{X}$, $k = 1, \ldots, m$, while C was set to $\widetilde{Z} + \sum_{k=1}^m \tilde{y}_k A_k$. We solved the problems using a primal-dual interior-point method (see [AHO96]).

Each row of Table 1 shows the number of times the rank r of X was achieved for the various values of m in the first experiment. The entries

m	1	2	3	4	5	6	7	8	9	10	11	12	13
10	22	766	212	0									
20	0	19	653	325	3								
30		0	51	635	314	0	0						
40			0	73	745	180	2	0					
50			0	1	161	709	129	0	0				
60				0	3	280	666	51	0	0			
70				0	0	9	486	492	13	0	0		
80					0	0	55	670	271	4	0	0	
90					0	0	0	160	712	128	0	0	
100						0	0	5	345	595	55	0	0

TABLE 1. Number of Occurrences of Rank(X) in 1000 Randomly Generated Problems with $n = 20$

m	Bounds on Rank(X)	Bounds on Rank(Z)
10	$1 \leq r \leq 4$	$16 \leq s \leq 19$
20	$1 \leq r \leq 5$	$15 \leq s \leq 19$
30	$2 \leq r \leq 7$	$13 \leq s \leq 18$
40	$3 \leq r \leq 8$	$12 \leq s \leq 17$
50	$3 \leq r \leq 9$	$11 \leq s \leq 17$
60	$4 \leq r \leq 10$	$10 \leq s \leq 16$
70	$4 \leq r \leq 11$	$9 \leq s \leq 16$
80	$5 \leq r \leq 12$	$8 \leq s \leq 15$
90	$5 \leq r \leq 12$	$8 \leq s \leq 15$
100	$6 \leq r \leq 13$	$7 \leq s \leq 14$

TABLE 2. Generic Bounds on Ranks for $n = 20$

corresponding to values of r falling outside of the range (22) are left blank. Table 2 gives the bounds (22) and (23) for the relevant values of m. The results are consistent with the generic property of nondegeneracy. They also show clearly that values of r in the center of its range are much more likely to occur than values equal to the bounds.

We close by noting that the issues of primal and dual nondegeneracy are fundamental to the analysis of convergence rates of primal-dual interior-point methods for SDP. These issues are discussed in [AHO96].

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