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SOME REGULARITY RESULTS FOR THE PSEUDOSPECTRAL ABSCISSA AND PSEUDOSPECTRAL RADIUS OF A MATRIX*

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Abstract. The ε-pseudospectral abscissa α_ε and radius ρ_ε of an $n \times n$ matrix are, respectively, the maximal real part and the maximal modulus of points in its ε-pseudospectrum, defined using the spectral norm. It was proved in [A.S. Lewis and C.H.J. Pang, *SIAM J. Optim.*, 19 (2008), pp. 1048–1072] that for fixed $\varepsilon > 0$, α_ε and ρ_ε are Lipschitz continuous at a matrix A except when α_ε and ρ_ε are attained at a critical point of the norm of the resolvent (in the nonsmooth sense), and it was conjectured that the points where α_ε and ρ_ε are attained are not resolvent-critical. We prove this conjecture, which leads to the new result that α_ε and ρ_ε are Lipschitz continuous, and also establishes the Aubin property with respect to both ε and A of the ε-pseudospectrum for the points $z \in \mathbb{C}$ where α_ε and ρ_ε are attained. Finally, we give a proof showing that the pseudospectrum can never be "pointed outwards."

Key words. pseudospectrum, pseudospectral abscissa, pseudospectral radius, eigenvalue perturbation, Lipschitz multifunction, Aubin property

AMS subject classifications. Primary, 15A18, 93D09, 49J52; Secondary, 65K10, 90C26

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1. Introduction.

Let $\|\cdot\|$ denote the vector or matrix 2-norm (spectral norm). For real $\varepsilon \geq 0$, the *ε-pseudospectrum* of a matrix $A \in \mathbb{C}^{n\times n}$ [TE05] is the union of the spectra of nearby matrices,

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : z \in \Lambda(A+E) \text{ for some } E \in \mathbb{C}^{n\times n} \text{ with } \|E\| \leq \varepsilon\}, \tag{1.1}$$

where $\Lambda(A)$ denotes the spectrum (set of eigenvalues) of A. Equivalently, Λ_ε is the upper level set of the norm of the resolvent of $A - zI$,

$$\Lambda_\varepsilon(A) = \left\{ z : \|(A - zI)^{-1}\| \geq \frac{1}{\varepsilon} \right\} \tag{1.2}$$

and the lower level set of the smallest singular value of $A - zI$,

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : \sigma_n(A - zI) \leq \varepsilon\}. \tag{1.3}$$

The *ε-pseudospectral abscissa* of A is the largest of the real parts of the elements of the pseudospectrum, i.e.,

$$\alpha_\varepsilon(A) = \max\{\text{Re } z : z \in \Lambda_\varepsilon(A)\}. \tag{1.4}$$

The case $\varepsilon = 0$ reduces to the spectral abscissa $\alpha(A)$, which measures the growth or decay of solutions to the continuous-time dynamical system $\dot{x} = Ax$; in particular, $\alpha(A)$ is negative if and only if the solution decays to zero for all initial states. For $\varepsilon > 0$, the pseudospectral abscissa of A characterizes asymptotic behavior when A

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is subject to perturbations bounded in norm by ε. Furthermore, as ε varies from 0 to ∞, the map α_ε ranges from measuring asymptotic behavior to measuring initial behavior of the solutions to $\dot{x} = Ax$ [BLO03, p. 86].

The analogous measure for discrete-time systems $x_{k+1} = Ax_k$ is the ε-*pseudospectral radius*

$$\rho_\varepsilon(A) = \max\{|z| : z \in \Lambda_\varepsilon(A)\}.$$

The case $\varepsilon = 0$ reduces to $\rho(A)$, the spectral radius of A, which is less than one if and only if solutions decay to zero for all initial states.

Below, we will refer to points where α_ε or ρ_ε is attained. By this we mean the points $z \in \Lambda_\varepsilon$ where the real part or the modulus, respectively, is maximized.

For fixed ε, $\Lambda_\varepsilon : A \to \Lambda_\varepsilon(A)$ is continuous [Kar03, Theorem 2.3.7]. Since Λ_ε is set-valued, continuity is to be understood in the Hausdorff metric. Recently, Lewis and Pang [LP08] proved that Λ_ε has further regularity properties. Specifically, they showed that Λ_ε has a local Lipschitz property known as the Aubin property everywhere except at *resolvent-critical* points (to be defined in the next section). It was also proved that for fixed $\varepsilon > 0$, α_ε (respectively, ρ_ε) is Lipschitz continuous at a matrix A if the points where α_ε (respectively, ρ_ε) is attained are not resolvent-critical (a consequence of [LP08, Corollary 7.2 and Theorem 5.2]). The fact that for a fixed matrix A the number of resolvent-critical points is finite leads to the property that Λ_ε, α_ε, and ρ_ε are Lipschitz around a given matrix A for all but finitely many $\varepsilon > 0$ [LP08, Corollary 8.5]. It was conjectured that the points where α_ε is attained are not resolvent-critical [LP08, Conjecture 8.9]. We prove this conjecture, which implies that for fixed $\varepsilon > 0$, α_ε is locally Lipschitz continuous on $\mathbb{C}^{n\times n}$. Our proof also applies to ρ_ε, proving that it is also locally Lipschitz. We also prove the Aubin property of the ε-pseudospectrum with respect to both ε and A for the points $z \in \mathbb{C}$ where α_ε and ρ_ε are attained. Finally, we give a proof showing that Λ_ε can never be "pointed outwards."

2. Previous results and notation. Before stating the conjecture, we need the following known results and definitions from [LP08]. We write MSV : $M^n \rightrightarrows \mathbb{C}^n \times \mathbb{C}^n$, with

$$\text{MSV}(A) := \{(u, v) \mid u, v \text{ minimal left and right singular vectors of } A\}.$$

In this definition, u, v are minimal left and right singular vectors of A if they are unit vectors satisfying

$$\sigma_n(A)u = Av \quad \text{and} \quad \sigma_n(A)v = A^*u,$$

where A^* is the Hermitian transpose of A. The set

$$Y(A) := \{v^*u \mid (u, v) \in \text{MSV}(A)\}$$

will be a key tool in our analysis since, for a fixed A and for $z \notin \Lambda(A)$, we have [LP08, Proposition 4.5]

$$Y(A - zI) = \partial(-\sigma_n(A - zI)), \tag{2.1}$$

where ∂ is the subdifferential in the sense of [RW98, Definition 8.3]. This leads to the following.

Definition 2.1. *A point $z \in \mathbb{C}$ is resolvent-critical for $A \in \mathbb{C}^{n\times n}$ if either $z \in \Lambda(A)$ or $0 \in Y(A - zI)$.*

Thus, a resolvent-critical point is either an eigenvalue of A or a critical point of the norm of the resolvent in the nonsmooth sense (see [LP08, Proposition 4.7 and Definition 4.8]).

Now we are ready to state Lewis and Pang's conjecture.

Conjecture 2.1 (see [LP08, Conjecture 8.9]). *The points $z \in \Lambda_\varepsilon(A)$ where the pseudospectral abscissa $\alpha_\varepsilon(A)$ is attained are not resolvent-critical.*

In the following, we will also need the Aubin property, a local Lipschitz property for set-valued mappings.

Definition 2.2 (see [RW98, Definition 9.36]). *A mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ has the Aubin property at $\bar{x}$ for $\bar{u}$, where $\bar{x} \in \mathbb{R}^n$ and $\bar{u} \in S(\bar{x})$, if gph S is locally closed at $(\bar{x}, \bar{u})$ and there are neighborhoods V of $\bar{x}$ and W of $\bar{u}$, and a constant $\kappa \in \mathbb{R}_+$, such that*

$$S(x') \cap W \subset S(x) + \kappa|x' - x|\mathbb{B} \quad \text{for all } x, x' \in V,$$

where $\mathbb{B}$ is the unit ball in $\mathbb{R}^m$.

3. New results. Let bd $\Lambda_\varepsilon(A)$ denote the boundary of the pseudospectrum of A. We now state our main result on the resolvent-critical points of bd $\Lambda_\varepsilon(A)$, which is based on a result in [ABBO10].

Theorem 3.1. *If $z \in \operatorname{bd} \Lambda_\varepsilon(A)$ is resolvent-critical for some $\varepsilon > 0$ and a matrix A, then there exists an integer $m \geq 2$, $\tilde{\theta}$ real, and $\tilde{\rho}$ positive real such that for all $\omega < \pi/m$, Λ_ε contains m equally spaced circular sectors of angle at least ω centered at z, that is,*

$$\Lambda_\varepsilon(A) \supset \{z + \rho e^{i\theta} \mid \theta \in [\tilde{\theta} + 2\pi k/m - \omega/2, \tilde{\theta} + 2\pi k/m + \omega/2], \rho \leq \tilde{\rho}\}$$

for all $k = 0, 1, 2, \ldots, m - 1$.

Proof. Assume that $z \in$ bd $\Lambda_\varepsilon(A)$ is resolvent-critical. Since $\varepsilon > 0$, $z \notin \Lambda(A)$, so there exists a pair of singular vectors $(\tilde{u}, \tilde{v}) \in \mathrm{MSV}(A - zI)$ such that $\tilde{v}^*\tilde{u} = 0$. If the smallest singular value of $A - zI$ is simple, then it follows from [ABBO10, Theorem 9 and subsequent discussion] that there exists an integer $m \geq 2$ such that for all $\omega < \pi/m$, Λ_ε contains m circular sectors of angle at least ω centered at z as stated, and so the result is proved. If the smallest singular value of $A - zI$ is not simple, consider a perturbed matrix $A_\delta = A - \delta\tilde{u}\tilde{v}^*$ for $\delta \in (0, \varepsilon)$. Then, the smallest singular value of $A_\delta - zI$ is simple with value $\varepsilon - \delta$ and corresponding singular vectors $\tilde{u}, \tilde{v}$ with $\tilde{u}^*\tilde{v} = 0$. Thus, we can apply [ABBO10, Theorem 9] to A_δ, finding that for all $\omega < \pi/m$, $\Lambda_{\varepsilon-\delta}(A_\delta)$ contains $m \geq 2$ circular sectors of angle at least ω centered at z. But immediately from the definition, using the triangle inequality for the norm, we have (see Figure 3.1)

$$\Lambda_\varepsilon(A) \supset \Lambda_{\varepsilon-\delta}(A_\delta),$$

proving the result. ∎

We conjecture that the only possible value for m in Theorem 3.1 is 2. See [ABBO10, Figure 3].

Clearly, at a point where the pseudospectral abscissa or pseudospectral radius is attained, the pseudospectrum cannot contain $m \geq 2$ circular sectors as defined above. As a consequence, we have the following.

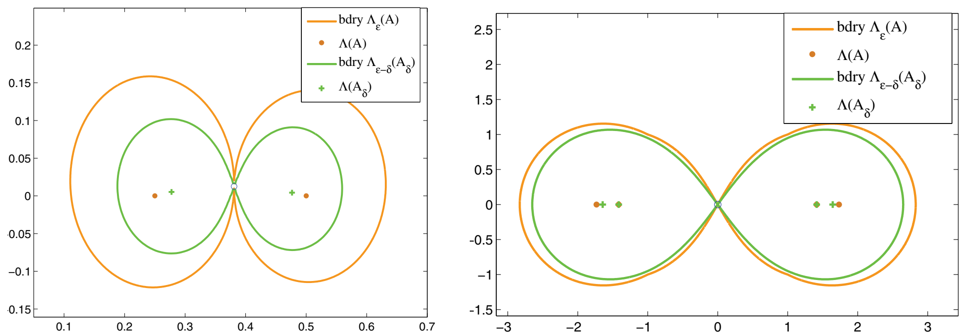

FIG. 3.1. *Figure illustrates the inclusion* $\Lambda_{\varepsilon-\delta}(A_\delta) \subset \Lambda_\varepsilon(A)$. *On the left,* A *is the* 4×4 *matrix with tangential coalescence given in* [ABBO10, *right panel of Figure* 1] *with* $\varepsilon = 0.0136$ *and* $\delta = 0.005$. *On the right,* A *is the reverse diagonal matrix with entries* 1, 1, 3, *and* 2 *(Gracia's example),* $\varepsilon = 1$ *and* $\delta = 0.1$. *The smallest singular value of* $A - zI$ *has multiplicity* 2 *in both cases, and* $m = 2$ *in both cases. The plots are obtained with the software package EigTool* [Wri02].

COROLLARY 3.2. *For any* $\varepsilon > 0$, *the points where the pseudospectral abscissa* α_ε *or pseudospectral radius* ρ_ε *is attained are not resolvent-critical.*

Thus, Conjecture 2.1 is proved. Furthermore, Theorem 3.1 implies the following regularity results about $\alpha_\varepsilon, \rho_\varepsilon$, and Λ_ε.

COROLLARY 3.3. *Let* $\varepsilon > 0$ *be given, and let* $z_* \in \operatorname{bd}\Lambda_\varepsilon(A_*)$ *be a point where the pseudospectral abscissa* $\alpha_\varepsilon(A_*)$ *or pseudospectral radius* $\rho_\varepsilon(A_*)$ *is attained for some matrix* A_*. *Then, the map* $A \to \Lambda_\varepsilon(A)$ *has the Aubin property at* A_* *for* z_*.

Proof. By Corollary 3.2, z_* is not resolvent-critical. The result follows from [LP08, Theorem 5.2]. ∎

COROLLARY 3.4. *For any fixed* $\varepsilon > 0$, α_ε *and* ρ_ε *are Lipschitz continuous at any matrix* A.

Proof. Let $A \in \mathbb{C}^{n\times n}$ be given. By Corollary 3.3, Λ_ε has the Aubin property at A at all the points where the pseudospectral abscissa or pseudospectral radius is attained. An application of [LP08, Corollary 7.2(a)] with $F = \Lambda_\varepsilon$, $g(x) = \operatorname{Re}(-x)$ proves the Lipschitz continuity of α_ε while using $F = \Lambda_\varepsilon$, $g(x) = -|x|$ proves the Lipschitz continuity of ρ_ε. ∎

COROLLARY 3.5. *Let* $z_* \in \mathbb{C}$ *be a point where the pseudospectral abscissa* $\alpha_{\varepsilon_*}(A)$ *or pseudospectral radius* $\rho_{\varepsilon_*}(A)$ *is attained for some* $\varepsilon_* > 0$ *and* $A \in \mathbb{C}^{n\times n}$. *Then the map* $\varepsilon \to \Lambda_\varepsilon(A)$ *has the Aubin property at* ε_* *for* z_*.

Proof. From (2.1) and Corollary 3.2, we have $0 \notin Y(A - z_*I) = \partial(-\sigma_n(A - z_*I))$. The result then follows from [LP08, Proposition 5.3], using, as is done there, the inclusion $-\partial(\sigma_n(A - zI)) \subset \partial(-\sigma_n(A - zI))$. ∎

In the terminology of [BLO03, Definition 4.5 and its corrigendum], a point z is called nondegenerate with respect to $\Lambda_\varepsilon(A)$ if $Y(A - zI) \neq \{0\}$. Thus, Corollary 3.2 implies that a point where the pseudospectral abscissa or pseudospectral radius is attained is nondegenerate. This leads to the following generalization of [BLO03, Proposition 4.8].

PROPOSITION 3.6. *Let* z_* *be a point where* $\alpha_\varepsilon(A)$ *or* $\rho_\varepsilon(A)$ *is attained for some* $\varepsilon > 0$ *and a matrix* A. *Then the boundary of* $\Lambda_\varepsilon(A)$ *is differentiable at* z_*; *i.e., the boundary of* $\Lambda_\varepsilon(A)$ *around* z_* *is a curve that is differentiable at* z_*.

Proof. This follows from [BLO03, Proposition 4.8] and the fact that z_* is nondegenerate. ∎

It was proved in [BLO03, Proposition 4.8] that the pseudospectrum cannot be "pointed outwards" at nondegenerate points. By this, one means that around a nondegenerate point z_*, the pseudospectrum can never be contained in a circular sector of angle strictly less than π centered at z_*. It was further stated that a more detailed analysis due to Trefethen shows that the pseudospectrum is never pointed outwards. Since the latter result, based on eigenvalue perturbation theory, was never published, we give a new proof here.

PROPOSITION 3.7. *Let z_* be on the boundary of the pseudospectrum, i.e., $z_* \in \operatorname{bd} \Lambda_\varepsilon(A)$ for some $\varepsilon > 0$ and a matrix A. The pseudospectrum near z_* cannot be contained in a circular sector of angle $< \pi$ centered at z_*; that is, for all $\omega \in [0, \pi)$, $\tilde{\theta} \in [0, 2\pi)$ and $\tilde{\rho}$ positive real, there exists a point $y \in \Lambda_\epsilon(A)$ such that we have $|y - z_*| \leq \tilde{\rho}$ but y is not contained in the circular sector*

$$z_* + \{\rho e^{i\theta} \mid [\tilde{\theta} - \omega/2, \tilde{\theta} + \omega/2], \rho \leq \tilde{\rho}\}.$$

Proof. If z_* is nondegenerate, then the result follows from [BLO03, Proposition 4.7]. Otherwise, z_* is resolvent-critical and the result follows from Theorem 3.1. ◻

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