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Conditioning of semidefinite programs

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Abstract. This paper studies the conditioning of semidefinite programs by analyzing the effect of small perturbations in problem data on the solution. Under the assumptions of strict complementarity and non-degeneracy, an explicit bound on the change in the solution is derived in a primal-dual framework, using tools from the Kantorovič theory. This approach also quantifies the size of permissible perturbations. We include a discussion of these results for block diagonal semidefinite programs, of which linear programming is a special case.

Key words. semidefinite programming – perturbation theory – Kantorovič theory – condition number

1. Introduction and notation

Our aim is to study the conditioning of semidefinite programs (SDP) with respect to small perturbations, *i.e.* to quantify the change in the solution of a semidefinite program induced by a sufficiently small perturbation in the problem data.

Let $\mathcal{S}^n$ denote the space of real, symmetric $n \times n$ matrices. The usual inner product on this space, denoted by $\bullet$, is defined by $A \bullet B \;=\; \mathrm{trace}(AB) \;=\; \sum_{i,j} a_{ij} b_{ij}$. We consider semidefinite programs in the following standard form:

$$\min \quad C \bullet X \quad \text{s.t.} \quad A_k \bullet X \;=\; b_k, \quad k = 1, 2, \ldots, m\,; \quad X \succeq 0, \tag{1}$$

where C, A_k $(k = 1, 2, \ldots, m)$ and X all belong to $\mathcal{S}^n$, b_k's are scalars, and by $X \succeq 0$, we mean that X lies in the closed, convex cone of positive semidefinite matrices. SDP enjoys a duality theory akin to that for linear programming. The dual of (1) is:

$$\max \quad b^T y \quad \text{s.t.} \quad \sum_{k=1}^{m} y_k A_k \;+\; Z = C; \quad Z \succeq 0, \tag{2}$$

where $Z \in \mathcal{S}^n$ is a positive semidefinite dual slack variable. The following Assumptions apply throughout the paper.

Assumption 1. *The matrices* A_k $(k = 1, \ldots, m)$ *are linearly independent, i.e. they span an m–dimensional subspace of* $\mathcal{S}^n$.

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Assumption 2. *The Slater condition holds for both the primal and the dual programs, i.e. there exists a primal feasible X and a dual feasible (y, Z) with X and Z strictly positive definite.*

Under these assumptions, it is well known that (optimal) solutions[1] exist to both the primal and the dual problems, and that the (optimal) objective values of both the programs are equal. Thus, a triple (X_0, y_0, Z_0) solves (1) and (2) if and only if X_0 is primal feasible, (y_0, Z_0) is dual feasible and the complementarity condition $X_0 \bullet Z_0 = 0$ is satisfied. Since for positive semidefinite matrices X_0 and Z_0, $X_0 \bullet Z_0 = 0$ if and only if $X_0 Z_0 = 0$, the complementarity condition implies that X_0 and Z_0 commute, and hence share an orthonormal system of eigenvectors, say Q_0. Clearly, this results in $\mathrm{rank}(X_0) + \mathrm{rank}(Z_0) \le n$.

Definition 1. *A primal solution X_0 and a dual solution (y_0, Z_0) are said to satisfy strict complementarity if $rank(X) + rank(Z) = n$.*

Let us denote the eigenvalues of X_0 and of Z_0 by

$$\lambda_0 = [\lambda_0^1, \dots, \lambda_0^n]^T \ge 0 \quad \text{and} \quad \omega_0 = [\omega_0^1, \dots, \omega_0^n]^T \ge 0 \tag{3}$$

respectively. Writing the primal solution as $X_0 = Q_0 \mathrm{Diag}(\lambda_0) Q_0^T$ and the dual slack solution as $Z_0 = Q_0 \mathrm{Diag}(\omega_0) Q_0^T$, we can restate the complementarity condition $X_0 Z_0 = 0$ as $\lambda_0^T \omega_0 = 0$, and strict complementarity as $\lambda_0 + \omega_0 > 0$. We assume without loss of generality that the components of λ_0 (of ω_0) are arranged in nonincreasing (nondecreasing) order, *i.e.* $\lambda_0^1 \ge \dots \ge \lambda_0^n$ and $\omega_0^1 \le \dots \le \omega_0^n$.

For the sake of completeness, we introduce the nondegeneracy definitions from [1]. Although the nondegeneracy conditions in [1] are developed in terms of the tangent space to the positive semidefinite cone, an equivalent linear algebra characterization (proved in [1]) is more amenable to our treatment here, and we use this as the definition of nondegeneracy. Since we assume that strict complementarity holds, we give the nondegeneracy definitions under this assumption, simplifying the definitions slightly.

Definition 2. *Let X_0 and (y_0, Z_0) be primal and dual solutions respectively, satisfying strict complementarity. Further, let $r = rank(X_0)$ and let Q_0 be a matrix whose columns form a common set of orthonormal eigenvectors for X_0 and Z_0. Partition Q_0 into Q_0^1 and Q_0^2, the $n \times r$ and the $n \times (n - r)$ matrices corresponding to the nonzero and the zero eigenvalues of X_0 (i.e. the zero and the nonzero eigenvalues of Z_0) respectively. Then,*
(i) X_0 is said to be primal nondegenerate if the matrices

$$B_k = \begin{bmatrix} (Q_0^1)^T A_k Q_0^1 & (Q_0^1)^T A_k Q_0^2 \\ (Q_0^2)^T A_k Q_0^1 & 0 \end{bmatrix}, \quad k = 1, 2, \dots, m$$

are linearly independent in $\mathcal{S}^n$, and
(ii) (y_0, Z_0) is said to be dual nondegenerate if the matrices

$$D_k = (Q_0^1)^T A_k Q_0^1 \quad k = 1, 2, \dots, m$$

span $\mathcal{S}^r$, the space of symmetric $r \times r$ matrices.

[1] Henceforth, we will use *solution* to mean *optimal solution*.

We add the following third assumption.

Assumption 3. *The primal* (1) *and the dual* (2) *programs have solutions* X_0 *and* (y_0, Z_0) *satisfying strict complementarity, primal nondegeneracy and dual nondegeneracy.*[2]

These assumptions guarantee that the primal and the dual solutions are unique [1].

It is notationally convenient to handle symmetric $n \times n$ matrices by mapping them onto vectors of length $n(n+1)/2$, so let $\mathbf{vec} : \mathcal{S}^n \longrightarrow \Re^{n(n+1)/2}$ be an isometry,[3] *i.e.* for all $A, B \in \mathcal{S}^n$, we have $A \bullet B = (\mathbf{vec}\, A)^T (\mathbf{vec}\, B)$. Then, the primal and dual equality constraints can be written as

$$\mathbf{A}\,\mathbf{vec}\, X = b; \quad \mathbf{A}^T y + \mathbf{vec}\, Z = \mathbf{vec}\, C,$$

where $\mathbf{A} \in \Re^{m \times n(n+1)/2}$ is a matrix whose k^{th} row is $(\mathbf{vec}\, A_k)^T$, and $b = [b_1, \ldots, b_m]^T \in \Re^m$. Now, the optimality conditions reduce to:

$$\mathbf{A}\,\mathbf{vec}\, X = b\,; \quad X \succeq 0 \quad \text{(primal feasibility)} \tag{4}$$

$$\mathbf{A}^T y + \mathbf{vec}\, Z = \mathbf{vec}\, C\,; \quad Z \succeq 0 \quad \text{(dual feasibility)} \tag{5}$$

$$XZ = 0 \quad \text{(complementarity)}. \tag{6}$$

It is easy to show that for two symmetric, positive semidefinite matrices X_0 and Z_0, the condition $X_0 Z_0 = 0$ is equivalent to $X_0 Z_0 + Z_0 X_0 = 0$. Hence, solving (4) – (6) reduces to finding a root of the function

$$F(X, y, Z) \equiv \begin{pmatrix} \mathbf{A}\,\mathbf{vec}\, X - b \\ \mathbf{A}^T\, y + \mathbf{vec}\,(Z - C) \\ \frac{1}{2}\mathbf{vec}\,(XZ + ZX) \end{pmatrix} \tag{7}$$

such that $X \succeq 0$ and $Z \succeq 0$.

Let I denote the identity matrix (the order being evident from context), and let $\mathbf{mat} : \Re^{n(n+1)/2} \longrightarrow \mathcal{S}^n$ be the inverse of $\mathbf{vec}$. We use $\circledast$ to denote the symmetrized Kronecker product introduced in [2], *i.e.* given $M, N \in \mathcal{S}^n$, $M \circledast N$ denotes the linear operator whose action on a vector $h \in \Re^{n(n+1)/2}$ is given by

$$(M \circledast N)\, h = \frac{1}{2}\, \mathbf{vec}\, \left(M\, (\mathbf{mat}\; h)\, N + N\, (\mathbf{mat}\; h)\, M \right).$$

Regarding F as a map from $\Re^{n(n+1)+m}$ to itself, the Jacobian of F is easily seen to be

$$J(X, y, Z) = \begin{pmatrix} \mathbf{A} & 0 & 0 \\ 0 & \mathbf{A}^T & I \circledast I \\ Z \circledast I & 0 & X \circledast I \end{pmatrix}. \tag{8}$$

We conclude this section with some more notation. For any two vectors $x = [x^1, \ldots, x^n]^T$ and $y = [y^1, \ldots, y^m]^T$, the pair (x, y) is used to denote the vector

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[2] Future references to *nondegeneracy* will mean both primal and dual nondegeneracy.

[3] For instance, we could take **vec** to be the operator that stacks the columns in the lower triangular part of a matrix into a vector, multiplying the offdiagonal elements by $\sqrt{2}$.

$[x^1, \ldots, x^n, y^1, \ldots, y^m]^T$. Unless explicitly indicated otherwise, we use the Euclidean norm $\|\cdot\|$ for vectors, and the induced 2-norm for matrices. The Frobenius norm of a matrix is denoted by $\|\cdot\|_F$. For a real, symmetric matrix A, we have $\|A\|_F = \|\mathbf{vec}\, A\| = \sqrt{A \bullet A}$. We let $u = (X, y, Z)$ stand for an element in the solution space $\mathcal{S}^n \times \Re^m \times \mathcal{S}^n$ equipped with the norm

$$\|u\| = \|(\mathbf{vec}\, X, y, \mathbf{vec}\, Z)\| = \left(\|X\|_F^2 + \|y\|^2 + \|Z\|_F^2\right)^{1/2}.$$

We denote by $N(u, \rho)$, an open ball of radius ρ centered at u, and by $\overline{N(u, \rho)}$, its closure. By $\mathrm{Lip}_\gamma(N(u, \rho))$, we mean the class of all functions that are Lipschitz continuous in $N(u, \rho)$, γ being the Lipschitz constant using the 2-norm. We say that a function is *uniformly* Lipschitz continuous if it is Lipschitz continuous at every point in its domain with the same Lipschitz constant. Finally, we use the compact notation $[\mathbf{A}, b, C]$ to denote the SDP's in (1) and (2).

2. Perturbation analysis for SDP

The two classical, qualitative notions of stability for a general mathematical programming problem are stability with respect to the optimal value, and stability with respect to the solution set [6]. Our analysis quantifies the latter for an SDP satisfying the assumptions, by explicitly bounding the change in the solution for a sufficiently small perturbation in the problem data. Consider a perturbation of the problem parameters A_k, b, and C in (1). In what follows,

$$\widetilde{\mathbf{A}} = \mathbf{A} + \Delta\mathbf{A}, \quad \tilde{b} = b + \Delta b, \quad \text{and} \quad \widetilde{C} = C + \Delta C \tag{9}$$

all denote perturbations in the original problem (1). Here, ΔC is symmetric, and $\Delta\mathbf{A}$ is a matrix whose k^{th} row is $(\mathbf{vec}\, \Delta A_k)^T$, with ΔA_k symmetric. Correspondingly, (7) for the perturbed system becomes

$$\widetilde{F}(u) \equiv \widetilde{F}(X, y, Z) \equiv \begin{pmatrix} \widetilde{\mathbf{A}}\, \mathbf{vec}\, X - \tilde{b} \\ \widetilde{\mathbf{A}}^T y + \mathbf{vec}\,(Z - \widetilde{C}) \\ \frac{1}{2}\mathbf{vec}\,(XZ + ZX) \end{pmatrix} = 0. \tag{10}$$

and the Jacobian of $\widetilde{F}$ (see (8)) becomes

$$\widetilde{J}(u) = \begin{pmatrix} \widetilde{\mathbf{A}} & 0 & 0 \\ 0 & \widetilde{\mathbf{A}}^T & I \circledast I \\ Z \circledast I & 0 & X \circledast I \end{pmatrix}. \tag{11}$$

We denote the solution to the original problem by $u_0 = (X_0, y_0, Z_0)$ and the solution to the perturbed problem by $\tilde{u}_0 = (\widetilde{X}_0, \tilde{y}_0, \widetilde{Z}_0)$.

We now state (without proof) a finite dimensional version of the Kantorovič theorem, which is central to our perturbation analysis.

Theorem 1 ([5], Ch. XVIII, Theorem 6). *Let $\rho_0 > 0$, $u_0 \in \Re^p$, $G : \Re^p \longrightarrow \Re^p$, and assume that G is continuously differentiable in $N(u_0, \rho_0)$. Assume for a vector norm and the induced operator norm that the Jacobian $G' \in Lip_\gamma(N(u_0, \rho_0))$ with $G'(u_0)$ nonsingular, and let*

$$\beta = \left\| G'(u_0)^{-1} \right\|, \quad \eta = \left\| G'(u_0)^{-1}\, G(u_0) \right\|, \quad \alpha = \beta\gamma\eta, \quad \rho = \frac{1 - \sqrt{1 - 2\alpha}}{\beta\gamma}.$$

If (a) $\alpha \le \frac{1}{2}$, and (b) $\rho \le \rho_0$, then
(i) G has a unique zero, say $\tilde{u}_0$, in $\overline{N(u_0, \rho)}$, and
(ii) Newton's method with unit steps, started at u_0, converges to this unique zero $\tilde{u}_0$.

The following corollary is immediate.

Corollary 1. *Let the conditions of Theorem 1 be satisfied. If $\alpha < 1/2$, then $G'(\tilde{u}_0)$ is nonsingular.*

Proof. Since the conditions of Theorem 1 are satisfied, G must have a zero, say $\tilde{u}_0$, such that

$$\begin{aligned} \|\tilde{u}_0 - u_0\| \le \rho &= \frac{1 - \sqrt{1 - 2\alpha}}{\beta\gamma} && (12)\\ &\le \frac{2\alpha}{\beta\gamma} \quad \text{when } 0 \le \alpha \le 1/2 && (13)\\ &< \frac{1}{\beta\gamma} \quad \text{when } \alpha < 1/2 \end{aligned}$$

so that

$$\left\| G'(\tilde{u}_0) - G'(u_0) \right\| \le \gamma \, \|\tilde{u}_0 - u_0\| < \frac{1}{\beta} = \frac{1}{\left\| G'(u_0)^{-1} \right\|}.$$

The Banach Lemma (Lemma 5 in Appendix A) now implies that $G'(\tilde{u}_0)$ is nonsingular.

□

Next, we state two preliminary lemmas needed for the perturbation analysis.

Lemma 1 ([2], Theorem 1). *Let $[\mathbf{A}, b, C]$ define an SDP satisfying the Assumptions. Then, the Jacobian at the solution, $J(u_0)$, is nonsingular.*

See [2] for a proof. Conversely, it is also true that if an SDP has a solution u_0 such that $J(u_0)$ is nonsingular, then strict complementarity and nondegeneracy hold at u_0 [4].

Lemma 2. *Let $[\mathbf{A}, b, C]$ define any SDP, not necessarily satisfying the Assumptions. Then, the Jacobian $J(u)$ associated with it is uniformly Lipschitz continuous, with 1 being a global Lipschitz constant.*

Proof. Let $u_1 = (X_1, y_1, Z_1)$ be any fixed point in $\mathcal{S}^n \times \Re^m \times \mathcal{S}^n$ and let $v = (v^1, v^2, v^3) \in \Re^{n(n+1)+m}$, with $v^1, v^3 \in \Re^{n(n+1)/2}$. Then, for any $u_2 = (X_2, y_2, Z_2) \in \mathcal{S}^n \times \Re^m \times \mathcal{S}^n$, we have

$$\begin{aligned}
\|J(u_2) - J(u_1)\| &= \max_{\|v\|=1} \|\{(Z_2 - Z_1) \circledast I\} v_1 + \{(X_2 - X_1) \circledast I\} v_3\| \\
&\le \max_{\|v\|=1} \|\{(Z_2 - Z_1) \circledast I\} v_1\| + \|\{(X_2 - X_1) \circledast I\} v_3\| \\
&\le \max_{\|v^1\|=1} \|\{(Z_2 - Z_1) \circledast I\} v_1\| + \max_{\|v^3\|=1} \|\{(X_2 - X_1) \circledast I\} v_3\| \\
&= \|(Z_2 - Z_1) \circledast I\| + \|(X_2 - X_1) \circledast I\| \\
&= \|Z_2 - Z_1\| + \|X_2 - X_1\| \qquad \text{(from Lemma 3, Appendix A)} \\
&\le \|u_2 - u_1\|,
\end{aligned}$$

thus concluding the proof.

□

For an SDP $[\mathbf{A}, b, C]$ satisfying the Assumptions and whose solution is $u_0 = (X_0, y_0, Z_0)$, and for its perturbation given in (9), we define the following quantities which will be used in the next theorem.

$$\beta_0 := \left\| J(u_0)^{-1} \right\| \quad \text{(see (8))},$$

$$\beta_1 := \|K\| \quad \text{where } K \text{ consists of the first } m + \tfrac{n(n+1)}{2} \text{ columns of } J(u_0)^{-1},$$

and

$$\delta_0 := \min\left(\min_{1\le i\le n} \left\{ \lambda_0^i : \lambda_0^i > 0 \right\}, \min_{1\le i\le n} \left\{ \omega_0^i : \omega_0^i > 0 \right\} \right) \qquad \text{(see (3))}$$

Theorem 2. *Let u_0 be the primal–dual solution to the SDP $[\mathbf{A}, b, C]$ satisfying the Assumptions, and let $[\widetilde{\mathbf{A}}, \tilde{b}, \widetilde{C}] = [\mathbf{A} + \Delta\mathbf{A}, b + \Delta b, C + \Delta C]$. Let*

$$\epsilon_0 := \|\Delta\mathbf{A}\| \, \|(\mathbf{vec}\, X_0, y_0)\| + \|(\Delta b, \mathbf{vec}\, \Delta C)\| .$$

If

$$\|\Delta\mathbf{A}\| \le \frac{1}{2\beta_1}, \quad \textit{and} \tag{14}$$

$$\epsilon_0 < \min\left(\frac{\sigma - 1}{2\sigma^2 \beta_0 \beta_1}, \frac{\delta_0}{2\sigma\beta_1} \right) \quad \textit{for some } 1 < \sigma \le 2, \tag{15}$$

then
(i) the SDP defined by $[\widetilde{\mathbf{A}}, \tilde{b}, \widetilde{C}]$ has a solution, say $\tilde{u}_0$, which satisfies

$$\|\tilde{u}_0 - u_0\| \le \frac{\sigma\beta_1\epsilon_0}{1 - \beta_1 \|\Delta\mathbf{A}\|}, \tag{16}$$

(ii) the solution to $[\widetilde{\mathbf{A}}, \tilde{b}, \widetilde{C}]$ is unique.
(iii) Newton's method with unit steps applied to $\widetilde{F}$, started at u_0, converges to $\tilde{u}_0$ quadratically.

Proof. To prove (i), we proceed in two steps. First, we use Kantorovič theorem to show that $\widetilde{F}$ has a root $\tilde{u}_0$ that satisfies the bound in (16). Second, we show that this root satisfies the positive semidefiniteness constraint, and hence is a solution to the SDP.

To use the Kantorovič theorem in the first step, we note the nonsingularity of the Jacobian $J(u_0)$ and the Lipschitz continuity of $J(\cdot)$ with Lipschitz constant $\gamma = 1$ (Lemma 1 and Lemma 2). Since

$$\Delta J := \widetilde{J}(u_0) - J(u_0) = \begin{bmatrix} \Delta \mathbf{A} & 0 & 0 \\ 0 & \Delta \mathbf{A}^T & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{17}$$

we have

$$\left\| J(u_0)^{-1} \Delta J \right\| \leq \beta_1 \, \|\Delta A\| \leq \frac{1}{2} \quad \text{(from (14))} \tag{18}$$

so that by the Banach Lemma (Lemma 5, Appendix A), $\widetilde{J}(u_0)$ is nonsingular with

$$\beta = \left\| \widetilde{J}(u_0)^{-1} \right\| \leq 2\beta_0. \tag{19}$$

Let

$$\eta = \left\| \widetilde{J}(u_0)^{-1} \widetilde{F}(u_0) \right\| \quad \text{and} \quad \alpha = \beta\eta. \tag{20}$$

We need only verify assumption (a) of Theorem 1, *i.e.* that $\alpha \leq \frac{1}{2}$; assumption (b) then follows trivially from the fact that the Lipschitz constant is global. We have

$$\widetilde{F}(u_0) = \begin{bmatrix} (\mathbf{A} + \Delta \mathbf{A}) \mathbf{vec}\, X_0 - (b + \Delta b) \\ (\mathbf{A} + \Delta \mathbf{A})^T y_0 + \mathbf{vec}\, Z_0 - \mathbf{vec}\, (C + \Delta C) \\ \frac{1}{2} \mathbf{vec}\, (X_0 Z_0 + Z_0 X_0) \end{bmatrix}$$

$$= \begin{bmatrix} (\Delta \mathbf{A}) \mathbf{vec}\, X_0 - \Delta b \\ (\Delta \mathbf{A})^T y_0 - \mathbf{vec}\, \Delta C \\ 0 \end{bmatrix}, \tag{21}$$

so that

$$\left\| J(u_0)^{-1} \widetilde{F}(u_0) \right\| \;\leq\; \beta_1 \left(\|\Delta \mathbf{A}\| \; \| (\mathbf{vec}\, X_0, y_0) \| \;+\; \|(\Delta b, \mathbf{vec}\, \Delta C)\| \right) = \beta_1 \epsilon_0. \tag{22}$$

Therefore, we obtain the estimate

$$\begin{aligned} \eta &= \left\| \widetilde{J}(u_0)^{-1} \widetilde{F}(u_0) \right\| \\ &= \left\| \left(I + J(u_0)^{-1} \Delta J \right)^{-1} J(u_0)^{-1} \widetilde{F}(u_0) \right\| \\ &\leq \frac{\beta_1 \epsilon_0}{1 - \left\| J(u_0)^{-1} \Delta J \right\|} \quad \text{(from (22) and Lemma 5, Appendix A)} \qquad (23) \\ &\leq 2\beta_1 \epsilon_0 \quad \text{(from (18))} \qquad (24) \end{aligned}$$

and from (19), (24) and (15), we conclude that

$$\alpha = \beta\eta \quad \le \quad 4\beta_0\beta_1\epsilon_0 \quad < \quad \frac{2(\sigma-1)}{\sigma^2} \quad \le \quad \frac{1}{2}. \tag{25}$$

Since $\alpha < \frac{1}{2}$, the hypotheses of the Kantorovič theorem hold, whence we can conclude that $\widetilde{F}$ has a unique zero $\tilde{u}_0$, with

$$\|u_0 - \tilde{u}_0\| \le \frac{1-\sqrt{1-2\alpha}}{\beta}. \tag{26}$$

We have

$$\begin{aligned} &\sigma^2\alpha - 2\sigma + 2 < 0 \quad \text{(from (25))} \\ \Rightarrow\ &\sigma^2\alpha^2 - 2\sigma\alpha + 2\alpha \le 0 \quad \text{(since } \alpha = \beta\eta \ge 0) \\ \Rightarrow\ &1 - \sigma\alpha \le \sqrt{1-2\alpha}, \end{aligned}$$

or equivalently,

$$\frac{1-\sqrt{1-2\alpha}}{\beta} \le \frac{\sigma\alpha}{\beta} = \sigma\eta,$$

so that, using (26),

$$\|u_0 - \tilde{u}_0\| \le \sigma\eta. \tag{27}$$

Combining this with (23) and (18) yields (16).

To show that this root is actually a solution to the SDP, we need to establish that $\widetilde{X}_0 \succeq 0$ and $\widetilde{Z}_0 \succeq 0$. To see this, note that

$$\begin{aligned} \left(\left\|\widetilde{X}_0 - X_0\right\|_F^2 + \|\tilde{y}_0 - y_0\|^2 + \left\|\widetilde{Z}_0 - Z_0\right\|_F^2\right)^{\frac{1}{2}} &= \|\tilde{u}_0 - u_0\| \\ &\le 2\sigma\beta_1\epsilon_0 \quad \text{(from (27) and (24))} \\ &< \delta_0 \quad \text{(from (15))} \end{aligned}$$

so that

$$\left\|\widetilde{X}_0 - X_0\right\| < \delta_0 \quad \text{and} \quad \left\|\widetilde{Z}_0 - Z_0\right\| < \delta_0. \tag{28}$$

Recalling that λ_0 and ω_0 were defined in (3), let $\tilde{\lambda}_0$ ($\tilde{\omega}_0$) be the vector of eigenvalues of $\widetilde{X}_0$ (of $\widetilde{Z}_0$), arranged in nonincreasing (nondecreasing) order. The following argument shows that $\widetilde{X}_0 \succeq 0$. For any $1 \le j \le n$,

$$\lambda_0^j > 0 \Rightarrow \tilde{\lambda}_0^j > 0 \qquad \text{(from (28) and Lemma 4, Appendix A)}$$

and

$$\begin{aligned} \lambda_0^j = 0 \Rightarrow \omega_0^j > 0 &\qquad \text{(strict complementarity of } X_0 \text{ and } Z_0) \\ \Rightarrow \tilde{\omega}_0^j > 0 &\qquad \text{(from (28) and Lemma 4, Appendix A)} \\ \Rightarrow \tilde{\lambda}_0^j = 0 &\qquad \text{(complementarity of } \widetilde{X}_0 \text{ and } \widetilde{Z}_0). \end{aligned}$$

A similar argument shows that $\widetilde{Z}_0 \succeq 0$. Thus, $\tilde{u}_0 = (\widetilde{X}_0, \tilde{y}_0, \widetilde{Z}_0)$ is indeed a solution to the perturbed SDP. This concludes the proof of (i) in the theorem.

The proof of (ii) in the theorem is an immediate consequence of Corollary 1: since $\widetilde{J}(\tilde{u}_0)$ is nonsingular, the perturbed problem $[\widetilde{\mathbf{A}}, \tilde{b}, \widetilde{C}]$ also satisfies strict complementarity and nondegeneracy [4], which in turn guarantees that the solution $\tilde{u}_0$ is unique [1].

The proof of (iii) is a consequence of the second conclusion of Theorem 1, combined with the nonsingularity of $\widetilde{J}(\tilde{u}_0)$.

□

See [2] for more on Newton's method in this context.

The following corollary establishes a bound on the *relative* error in the solution of a perturbed SDP, and thus introduces the notion of a *condition number* for semidefinite programs.

Corollary 2. *Let the conditions of Theorem 2 hold, and let $\Delta u_0 = \tilde{u}_0 - u_0$. Then,*

$$\frac{\|\Delta u_0\|}{\|u_0\|} \leq \frac{\sigma\ \|K\|\ \|L\|}{1 - \|K\|\,\|\Delta\mathbf{A}\|}\left(\frac{\|\Delta\mathbf{A}\|\ \|(\mathbf{vec}\, X_0, y_0)\|}{\|(b, \mathbf{vec}\, C)\|} + \frac{\|(\Delta b, \mathbf{vec}\,\Delta C)\|}{\|(b, \mathbf{vec}\, C)\|}\right), \tag{29}$$

where K (respectively L) consists of the first $m + n(n+1)/2$ columns (respectively rows) of $J(u_0)^{-1}$ (respectively $J(u_0)$).

Proof. Observe that u_0 satisfies $Lu_0 = (b, \mathbf{vec}\, C)$, so that

$$\|L\|\,\|u_0\| \geq \|(b, \mathbf{vec}\, C)\|\,.$$

The result follows by combining this inequality with (16).

□

Thus, $\sigma\ \|K\|\ \|L\|$ may be viewed as a condition number. In the special case $\Delta\mathbf{A} = 0$, we have $\beta = \beta_0$, the inequality in (15) can be relaxed to

$$\epsilon_0 < \min\left(\frac{2(\sigma - 1)}{\sigma^2 \beta_0 \beta_1}, \frac{\delta_0}{\sigma\beta_1}\right) \quad (1 < \sigma \leq 2)$$

and (29) reduces to

$$\frac{\|\Delta u_0\|}{\|u_0\|} \leq \sigma\ \|K\|\ \|L\|\ \left(\frac{\|(\Delta b, \mathbf{vec}\,\Delta C)\|}{\|(b, \mathbf{vec}\, C)\|}\right).$$

3. Block diagonal SDP and linear programs

Several practical problems (for instance, linear matrix inequalities in control theory and problems from optimal structural design) have an inherent block diagonal structure, when formulated as semidefinite programs. In this section, we consider semidefinite programming over the space of real, symmetric, block diagonal matrices. Then, we discuss linear programming as a special case.

3.1. Block diagonal semidefinite programs

Given a positive integer vector $q = [q^1, \dots, q^p]$ with $n = \sum_{i=1}^p q^i$, let $\mathcal{B}^q$ denote the space of all real, symmetric, $n \times n$ block diagonal matrices whose i^{th} diagonal block is of size q^i. The dimension of this space is $g = \sum_{i=1}^p q^i(q^i + 1)/2$, and we define **vec** to be an isometry from $\mathcal{B}^q$ to $\Re^g$. We refer to the i^{th} diagonal block of a matrix $X \in \mathcal{B}^q$ as $X(i)$, and we use the notation $\prod_{i=1}^p X(i)$ to denote the matrix X. The primal and the dual semidefinite programs can be formulated over this space of block diagonal matrices, just as in (1) and (2), but with $\mathcal{S}^n$ replaced by $\mathcal{B}^q$.

The nondegeneracy conditions of [1] can be extended in a straightforward way to $\mathcal{B}^q$ via tangent and normal spaces to the cone of positive semidefinite matrices in $\mathcal{B}^q$ (see Appendix B). As in Section 1, we provide an equivalent linear algebra characterization of nondegeneracy here, assuming that strict complementarity holds.

Definition 3. *Let X_0 and (y_0, Z_0) be primal and dual solutions respectively, satisfying strict complementarity. Further, let $Q_0(i) = [Q_0^1(i) \;\; Q_0^2(i)]$ be a matrix whose columns form a set of orthonormal eigenvectors of $X_0(i)$ and $Z_0(i)$, with $Q_0^1(i)$ and $Q_0^2(i)$ corresponding to the nonzero and the zero eigenvalues of $X_0(i)$ (i.e. the zero and the nonzero eigenvalues of $Z_0(i)$) respectively. Then,*
(i) X_0 is said to be primal nondegenerate if the matrices

$$B_k = \prod_{i=1}^{p} \begin{bmatrix} Q_0^1(i)^T A_k(i) Q_0^1(i) & Q_0^1(i)^T A_k(i) Q_0^2(i) \\ Q_0^2(i)^T A_k(i) Q_0^1(i) & 0 \end{bmatrix}, \quad k = 1, 2, \dots, m$$

are linearly independent in $\mathcal{B}^q$, and
(ii) (y_0, Z_0) is said to be dual nondegenerate if

$$D_k = \prod_{i=1}^{p} Q_0^1(i)^T A_k(i) Q_0^1(i)\,, \quad k = 1, 2, \dots m$$

span $\mathcal{B}^r$, where $r^i = rank(X_0(i))$.

All the results in Section 2 hold verbatim in the block diagonal case, but with the understanding that the operator **vec**, its associated symmetrized Kronecker product $\circledast$, and the nondegeneracy conditions are interpreted as just described. The function F now maps $\Re^{2g+m}$ to itself, the Jacobian J has dimension $2g + m$, while K (respectively L) consists of the first $g + m$ columns (respectively rows) of $J(u_0)^{-1}$ (respectively $J(u_0)$).

3.2. Linear programming

In the case $q = [1, \dots, 1]$ with $g = n$, the A_k's, C and X are $n \times n$ diagonal matrices, and SDP over the space $\mathcal{B}^q$ reduces to linear programming (LP). It is interesting to see what our perturbation analysis for block diagonal SDP's yields for LP. Then $\mathbf{A} \in \Re^{m \times n}$ is the matrix whose k^{th} row is $(\mathbf{vec}\, A_k)^T$, and letting $c, x, z \in \Re^n$ stand for $\mathbf{vec}\, C$, $\mathbf{vec}\, X$ and $\mathbf{vec}\, Z$ respectively, we get the primal linear program

$$\min \quad c^T x \quad \text{s.t.} \quad \mathbf{A}x = b; \quad x \geq 0, \tag{30}$$

and its dual

$$\max \quad b^T y \quad \text{s.t.} \quad \mathbf{A}^T y + z = c; \quad z \geq 0. \tag{31}$$

Our assumptions here are the same as those in Section 1. Assumption 1 implies that $\mathbf{A}$ has full row rank. It can be verified (see Appendix C) that the nondegeneracy assumption (Assumption 3) implies that the primal solution, rearranged as $x_0 = (x_0^1, x_0^2)$, has exactly m strictly positive components (denoted by x_0^1), and, that if we rearrange the columns of $\mathbf{A}$ as $[\mathbf{A}_1 \;\; \mathbf{A}_2]$ with $\mathbf{A}_1$ and $\mathbf{A}_2$ corresponding to x_0^1 and x_0^2 respectively, then $\mathbf{A}_1$ is nonsingular. Writing $z_0 = (z_0^1, z_0^2)$ accordingly, we have $z_0^1 = 0$, and by strict complementarity, $z_0^2 > 0$. Therefore,

$$J(u_0) = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{A}_1^T & I & 0 \\ 0 & 0 & \mathbf{A}_2^T & 0 & I \\ 0 & 0 & 0 & \mathrm{Diag}(x_0) & 0 \\ 0 & \mathrm{Diag}(z_0^2) & 0 & 0 & 0 \end{bmatrix}.$$

Thus, Theorem 2 holds with $\beta_1 = \|K\|$, with

$$K = \begin{bmatrix} \mathbf{A}_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathbf{A}_1^{-T} & 0 \\ 0 & 0 & 0 \\ 0 & -\mathbf{A}_2\mathbf{A}_1^{-T} & I \end{bmatrix}.$$

However, under the same assumptions, it is possible to use a simple linear algebra argument[4] to obtain a perturbation bound. Rearranging c as (c^1, c^2) and defining

$$R = \begin{bmatrix} \mathbf{A}_1 & 0 & 0 \\ 0 & \mathbf{A}_1^T & 0 \\ 0 & \mathbf{A}_2^T & I \end{bmatrix}, \tag{32}$$

the solution to the LP is given by $R\,(x_0^1, y_0, z_0^2) = (b, c^1, c^2)$, and $x_0^2 = 0$, $z_0^1 = 0$. Since this holds for any sufficiently small perturbation (so that the basis does not change), the standard perturbation result for square, nonsingular linear systems [13, p. 26] gives

$$\begin{aligned} \|\Delta u_0\| &= \left\|(\Delta x_0^1, \Delta y_0, \Delta z_0^2)\right\| \\ &\leq \frac{\left\|R^{-1}\right\|}{1 - \left\|R^{-1}\right\| \|\Delta R\|} \left(\|\Delta R\| \;\; \|(x_0, y_0)\| + \|(\Delta b, \Delta c)\|\right). \end{aligned} \tag{33}$$

Here, ΔR is the matrix obtained by replacing $\mathbf{A}_1$, $\mathbf{A}_2$ and I in (32) by $\Delta\mathbf{A}_1$, $\Delta\mathbf{A}_2$ and 0 respectively. Since $\|\Delta R\| = \|\Delta A\|$ and $\left\|R^{-1}\right\| = \|K\|$, the bound obtained in Theorem 2 via the Kantorovič theory specializes, except for the factor of σ, to the one in (33) obtained by the linear algebra approach.

[4] The authors thank an anonymous referee for this observation.

4. Concluding remarks

An alternative way to formulate the problem is to introduce a perturbation parameter t (assumed to be a scalar, for simplicity), and study the solution $u(t) = (X(t), y(t), Z(t))$ of the parametrized SDP $[\mathbf{A}(t), b(t), C(t)]$, where $\mathbf{A}(\cdot)$, $b(\cdot)$ and $C(\cdot)$ are assumed to be at least $\mathcal{C}^1$, and $t = t_0$ corresponds to the original problem. We may now regard F defined in (7) to be $F(t, X, y, Z)$, and replace $\mathbf{A}$, b, C in the right hand side of (7) by the functions $\mathbf{A}(t)$, $b(t)$, $C(t)$ respectively. In view of Lemma 1, the implicit function theorem states that $u'(t)$ is well defined and continuous in some neighborhood $(t_0 - \epsilon_0, t_0 + \epsilon_0)$ around t_0, and allows us to compute the derivative of the solution at t_0 as

$$u'(t_0) = -J(u_0)^{-1} F'(t_0, X_0, y_0, Z_0), \tag{34}$$

where the prime notation ($'$) stands for the derivative with respect to t. Indeed, we can conclude that $\forall\, \delta > 0$, $\exists\, \epsilon(\delta) > 0$ such that

$$\|u(t) - u(t_0)\| \leq \left(\left\|u'(t_0)\right\| + \delta\right) \mid t - t_0 \mid \quad \forall t \in (t_0 - \epsilon(\delta), t_0 + \epsilon(\delta))$$

Thus, $\left\|u'(t_0)\right\|$ can be considered to be an asymptotic error bound.[5] However, the implicit function theorem does not provide a way to estimate $\epsilon(\delta)$. On the other hand, the Kantorovič approach uses $\Delta\mathbf{A} = \mathbf{A}(t) - \mathbf{A}(t_0)$, $\Delta b = b(t) - b(t_0)$ and $\Delta C = C(t) - C(t_0)$ to provide explicit bounds both on $\|\tilde{u}_0 - u_0\|$ (see (16)) and on the permissible perturbations (see (14) and (15)), without any assumptions on the functions $\mathbf{A}(\cdot)$, $b(\cdot)$, and $C(\cdot)$. In the limiting case $t \longrightarrow t_0$, we have $\epsilon_0 \longrightarrow 0$, so that we may let $\sigma \longrightarrow 1$ in (15). Then, from (20) and (21), the quotient $\eta / \mid t - t_0 \mid \longrightarrow \left\|u'(t_0)\right\|$, where $u'(t_0)$ is given in (34). Hence, the Kantorovič bound in (27) divided by $\mid t - t_0 \mid$ approaches $\left\|u'(t_0)\right\|$.

We now make a few remarks about the assumptions made. Assumption 1 is a mere convenience. If the A_k are linearly dependent, then the equality constraints are either inconsistent or redundant. Assumption 2 (the Slater condition) guarantees that the problem remains well–posed under small perturbations. A problem violating the Slater condition is ill–posed in the sense that it could become infeasible under an arbitrarily small perturbation. Assumption 3 (the nondegeneracy and strict complementarity condition), which guarantees a unique solution to the SDP, is crucial for the application of the Kantorovič theory. In the absence of further qualifications on the data and the nature of the perturbation, the solution set may not be outer semicontinuous [12, Def. 5.4] if the Slater condition is violated, and may not be inner semicontinuous [12, Def. 5.4] if the nondegeneracy condition is violated. Note that Assumption 3 is generically satisfied [1].

For linear programming, in the special case of perturbations to b alone (*i.e.* $\Delta\mathbf{A} = 0$, $\Delta c = 0$) and under the assumption that the perturbed problem has a nonempty solution set, Mangasarian and Shiau [7] bound the distance between the solution sets of the original and the perturbed problems in terms of the perturbation in b. Robinson [11] uses Hoffman's lemma for linear inequalities to bound the distance between the solution set of a linear program and a fixed point in the solution space. Renegar [9,10] introduces the notion of the distance to ill–posedness, and derives error bounds for a general class

[5] This notion was suggested by an anonymous referee.

of mathematical programs in the setting of reflexive Banach spaces. However, a feature common to all these results (including ours) is that they require some form of knowledge of the solution (or the active set at the solution) of the original program. In this sense, computing the condition number of an LP or SDP involves at least as much work as solving the program itself.

Appendices

A. Miscellaneous results

Lemma 3 ([2], Lemma 2). *For commuting real, symmetric matrices M and N, let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ denote the eigenvalues of M and N respectively, with $v_1, \ldots, v_n$ being a common basis of orthonormal eigenvectors. The $n(n+1)/2$ eigenvalues of $M \circledast N$ are given by*

$$\frac{1}{2}\left(\alpha_i \beta_j + \beta_i \alpha_j\right), \quad 1 \le i \le j \le n,$$

with the corresponding set of orthonormal eigenvectors

$$\left\{ \begin{array}{ll} \mathbf{vec}\,(v_i v_i^T) & \text{if } i = j \\ \frac{1}{\sqrt{2}}\mathbf{vec}\,(v_i v_j^T + v_j v_i^T) & \text{if } i < j \end{array} \right\}.$$

The proof is straightforward.

Lemma 4. *Let A and $A+E$ be real, symmetric matrices with eigenvalues $\lambda_1 \geq \ldots \lambda_n$ and $\mu_1 \geq \ldots \geq \mu_n$ respectively. Then*

$$|\lambda_i - \mu_i| \leq \|E\|, \quad i = 1, \ldots, n.$$

See, for instance, [8, p. 58] for a proof.

Lemma 5 (Banach Lemma). *Let A be a square nonsingular matrix and let $\widetilde{A} = A + E$ be a perturbation of A. If $\left\|A^{-1}E\right\| < 1$, then $\widetilde{A}$ is nonsingular, and*

$$\left\|\widetilde{A}^{-1}\right\| \leq \frac{\|A^{-1}\|}{1 - \left\|A^{-1}E\right\|}.$$

See [14, p. 118] for a proof.

B. Nondegeneracy for block diagonal SDP

Here, we extend the nondegeneracy definitions of [1] to the block diagonal case, without assuming strict complementarity.

Consider $\mathcal{B}^q$, the space of real, symmetric block diagonal matrices with block structure $q = [q^1, \ldots, q^p]$. Recall that we refer to the i^{th} diagonal block of a matrix $X \in \mathcal{B}^q$ by $X(i)$, and that we use the notation $\prod_{i=1}^{p} X(i)$ to denote the matrix X. Given a positive integer vector $r = [r^1, \ldots, r^p]$, let $\mathcal{M}_r$ be the smooth manifold of matrices in $\mathcal{B}^q$ whose i^{th} block is of rank r^i $(i = 1, \ldots, p)$. Consider SDP over the space $\mathcal{B}^q$, and let $X \in \mathcal{M}_r$ be primal feasible. Let $Q(i) = [Q^1(i) \quad Q^2(i)] \quad (i = 1, \ldots, p)$ be an orthonormal set of eigenvectors of $X(i)$, such that $Q^1(i) \in \Re^{q^i \times r^i}$ and $Q^2(i) \in \Re^{q^i \times (q^i - r^i)}$ are eigenvectors corresponding to the nonzero and the zero eigenvalues of $X(i)$ respectively. Then, the tangent space to $\mathcal{M}_r$ at X is given by [3]

$$\mathcal{T}_X(\mathcal{M}_r) = \left\{ \prod_{i=1}^{p} Q(i) \begin{bmatrix} U_i & V_i \\ V_i^T & 0 \end{bmatrix} Q(i)^T \ : \ U_i \in \mathcal{S}^{r^i}, \ V_i \in \Re^{r^i \times (q^i - r^i)} \right\}.$$

Definition 4. *$X \in \mathcal{B}^q$ is primal nondegenerate if it is primal feasible and $\mathcal{T}_X(\mathcal{M}_r) + \mathcal{N} = \mathcal{B}^q$, where $\mathcal{N}$ is the orthogonal complement (with respect to $\bullet$) of* $\mathrm{Span}(A_1, \ldots, A_m)$ *in $\mathcal{B}^q$.*

Similarly, let $s = [s^1, \ldots, s^p]$ be a positive integer vector, and let $\mathcal{M}_s$ be the smooth manifold of matrices in $\mathcal{B}^q$ whose i^{th} block is of rank s^i. Let (y, Z) be dual feasible with $Z \in \mathcal{M}_s$, and let $P(i) = [P^1(i) \quad P^2(i)]$ be an orthonormal set of eigenvectors of $Z(i)$ such that $P^1(i) \in \Re^{q^i \times (q^i - s^i)}$ and $P^2(i) \in \Re^{q^i \times s^i}$ are eigenvectors corresponding to the zero and the nonzero eigenvalues of $Z(i)$ respectively. Then, the tangent space to $\mathcal{M}_s$ at Z is given by

$$\mathcal{T}_Z(\mathcal{M}_s) = \left\{ \prod_{i=1}^{p} P(i) \begin{bmatrix} 0 & V_i \\ V_i^T & W_i \end{bmatrix} P(i)^T \ : \ V_i \in \Re^{(q^i - s^i) \times s^i}, \ W_i \in \mathcal{S}^{s^i} \right\}.$$

Definition 5. *$(y, Z) \in \Re^m \times \mathcal{B}^q$ is dual nondegenerate if it is dual feasible and* $\mathcal{T}_Z(\mathcal{M}_s) + \mathrm{Span}(A_1, \ldots, A_m) = \mathcal{B}^q$.

The following two theorems relate the nondegeneracy definitions given above with an equivalent linear algebra characterization. The proofs are along the same lines as in [1], and are omitted.

Theorem 3. *Let $X \in \mathcal{M}_r$ be primal feasible. If X is primal nondegenerate, then the following dimensionality condition necessarily holds:*

$$\sum_{i=1}^{p} (q^i - r^i)(q^i - r^i + 1)/2 \leq \left(\sum_{i=1}^{p} q^i (q^i + 1)/2 \right) - m.$$

Further, let $Q(i) = [Q^1(i) \ Q^2(i)]$ be as defined above, with $Q^1(i)$ and $Q^2(i)$ corresponding to the nonzero and zero eigenvalues of $X(i)$ respectively. Then, X is primal nondegenerate if and only if the matrices

$$B_k = \prod_{i=1}^{p} \begin{bmatrix} Q^1(i)^T A_k(i) Q^1(i) & Q^1(i)^T A_k(i) Q^2(i) \\ Q^2(i)^T A_k(i) Q^1(i) & 0 \end{bmatrix}, \quad k = 1, 2, \ldots, m$$

are linearly independent in $\mathcal{B}^q$.

Theorem 4. *Let (y, Z) be dual feasible with $Z \in \mathcal{M}_s$. If (y, Z) is dual nondegenerate, then the following dimensionality condition necessarily holds:*

$$\sum_{i=1}^{p}(q^i - s^i)(q^i - s^i + 1)/2 \le m.$$

Further, let $P(i) = [P^1(i)\ \ P^2(i)]$ be as defined above, with $P^1(i)$ and $P^2(i)$ corresponding to the zero and the nonzero eigenvalues of Z respectively. Then, Z is dual nondegenerate if and only if the matrices

$$D_k = \prod_{i=1}^{p}\left[\, P^1(i)^T A_k(i) P^1(i)\,\right], \quad k = 1, 2, \ldots m$$

span $\mathcal{B}^{q-s}$.

For the i^{th} block, if complementarity holds (implying that $X(i)$ and $Z(i)$ commute), we can choose (without loss of generality) $P(i) = Q(i)$, and if strict complementarity holds, we can choose $P^1(i) = Q^1(i)$ and $P^2(i) = Q^2(i)$.

C. Reduction of block diagonal nondegeneracy conditions to the LP case

Consider Definition 3 in the case of LP, where $q = [1, \ldots, 1]$, and for each $i = 1, \ldots, p$, one of $Q_0^1(i)$ and $Q_0^2(i)$ is the scalar 1, and the other is empty. Consequently, for each k, B_k is a diagonal matrix consisting of the entries in A_k corresponding to nonzero primal variables. Suppose there are r of these. The primal nondegeneracy condition (see Definition 3) requires the B_k $(k = 1, \ldots, m)$ to be linearly independent, and hence $r \ge m$. Furthermore, $D_k = B_k$, and the dual nondegeneracy condition requires the B_k $(k = 1, \ldots, m)$ to span the space of diagonal matrices of dimension r, and hence $m \ge r$. Thus $r = m$, and the nondegeneracy conditions reduce to the standard condition in LP, namely nonsingularity of the basis matrix.

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