Stability Theory for Dissipatively Perturbed Hamiltonian Systems

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Abstract

It is shown that for an appropriate class of dissipatively perturbed Hamiltonian systems, the number of unstable modes of the dynamics linearized at a nondegenerate equilibrium is determined solely by the index of the equilibrium regarded as a critical point of the Hamiltonian. In addition, the movement of the associated eigenvalues in the limit of vanishing dissipation is analyzed.

1. Introduction

An autonomous Hamiltonian system of ordinary differential equations has the form

$$\dot{x} = J(x)\,\nabla H(x), \tag{1.1}$$

where ∇H denotes the gradient of the Hamiltonian $H(x)$ with respect to the variable x, the matrix $J(x)$ is skew-symmetric for all x, and solutions $x(t)$ of (1.1) are curves in phase space. In the classic setting with n degrees of freedom, $x(t) \in \Re^{2n}$ and

$$J(x) = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \tag{1.2}$$

each identity block having dimension n. Although we need not assume $J(x)$ has this particular form, we shall assume throughout that $J(x)$ is nonsingular. Thus the equilibrium solutions of (1.1), i.e. trajectories satisfying $\dot{x}_e = 0$, are precisely the critical points of the Hamiltonian, i.e. those points in phase space satisfying

$$\nabla H(x_e) = 0. \tag{1.3}$$

We shall also assume that critical points are nondegenerate in the sense that $\nabla^2 H(x_e)$, the second variation of the Hamiltonian, is nonsingular. This assumption excludes interesting cases of relative equilibria where the singularity

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of $\nabla^2 H(x_e)$ is associated with an underlying symmetry of the dynamics. Such singular cases will be discussed in future work.

At an equilibrium point x_e there are two eigenvalue problems bearing upon the stability of the dynamical system. The first is a nonsymmetric problem, obtained by linearizing the dynamics (1.1) and separating out time:

$$JSu = \lambda u, \tag{1.4}$$

where $S = \nabla^2 H(x_e)$ and $J = J(x_e)$. The matrix JS is said to be Hamiltonian (or infinitesimally symplectic). Because of the special structure of JS, its eigenvalues have two-fold symmetry, i.e. they are symmetrically placed in the complex plane with respect to both the real and imaginary axes. If all eigenvalues are imaginary, the equilibrium point is said to be spectrally stable (or elliptic). Spectral stability is a necessary, though far from sufficient, condition for stability under the full nonlinear dynamics.

The second eigenvalue problem is the symmetric one associated with the second variation of the Hamiltonian:

$$Sv = \kappa v. \tag{1.5}$$

The connection between the two eigenvalue problems (1.5) and (1.4) is a classic topic of investigation, and there are some simple conclusions that are immediate. For example, if (1.5) has only positive eigenvalues, the matrix S has a real symmetric square root and JS is similar to the skew-symmetric matrix $S^{1/2}JS^{1/2}$, so (1.4) has only imaginary eigenvalues. However, it is possible that (1.4) has only imaginary eigenvalues while (1.5) has eigenvalues of both signs, which allows the possibility that spectral stability may occur at critical points of the Hamiltonian which are not minima. Thus, there is not a sharp correspondence between the two eigenvalue problems.

We shall show that there *is* a sharp correspondence between (1.5) and certain dissipative perturbations of (1.4). The perturbed eigenvalue problem is generated by adding damping to the original system (1.1). A mathematically convenient and physically sensible way to do this is to consider dynamics of the form

$$\dot{x} = (J(x) - \epsilon D(x))\nabla H(x), \tag{1.6}$$

where $D(x)$ is real, symmetric, positive semidefinite, and $\epsilon > 0$. Since $J(x)$ is nonsingular and skew-symmetric, $(J(x) - \epsilon D(x))$ is nonsingular for all ϵ. Consequently the two systems (1.1) and (1.6) have the same equilibrium points, namely the critical points of the Hamiltonian.

The eigenvalue problem obtained from linearizing (1.6) at an equilibrium x_e is

$$(J - \epsilon D)Su = \mu u, \tag{1.7}$$

where $D = D(x_e)$. Provided that the semidefinite dissipation matrix D satisfies the condition

$$z^*SDSz > 0, \quad \forall \text{ eigenvectors } z \text{ of } JS \text{ with pure imaginary eigenvalues,} \tag{1.8}$$

it will be shown that for all $\epsilon > 0$, *the number of eigenvalues of* (1.7) *in the right half-plane equals the number of negative eigenvalues of* (1.5). In this statement, we adopt the conventions that half-plane means strict half-plane, excluding the imaginary axis, and that any multiple eigenvalues are counted according to their algebraic multiplicity. Furthermore, we demonstrate that no eigenvalue of (1.7) can remain on the imaginary axis for $\epsilon > 0$, so the number in the left half-plane equals the number of positive eigenvalues of (1.5).

Our results imply that, given condition (1.8) and nondegeneracy of the critical point, *the number of exponentially growing modes of the linearized dynamics equals the index of the equilibrium regarded as a critical point of the Hamiltonian.* Furthermore, there are no polynomially growing modes. It follows that the *only* nondegenerate equilibria of (1.6) that are linearly stable are minima of the Hamiltonian. Any such equilibrium point is a nonlinearly stable solution of the undamped dynamics (1.1) by the Lagrange-Dirichlet Theorem [22], p. 208, and, for appropriate classes of dissipation $D(x)$, can be shown to be an asymptotically stable solution of the damped dynamics (1.6) by a Lyapunov argument.

The result just described is precisely stated in Theorem 1 below, and is proven by a simple homotopy connecting the two eigenvalue problems (1.5) and (1.7). However that argument does not explicitly describe the behavior of eigenvalues of (1.7) in the limit of vanishing dissipation as $\epsilon \to 0$. The bulk of our presentation is concerned with a perturbation analysis that addresses this issue in detail. If the eigenvalues of the unperturbed problem are all assumed to be simple, the perturbation analysis is straightforward, but here we deal with the case of arbitrary Jordan structure.

Our perturbation analysis also demonstrates that the definiteness hypothesis (1.8) is necessary in the sense that without this assumption the conclusion of Theorem 1 is false. In Section 9 we demonstrate that the framework (1.6) captures a wide class of systems arising in applications, namely Lagrangian dynamics subject to Rayleigh dissipation. For such systems hypothesis (1.8) can be verified without any explicit knowledge of the eigenvectors.

While we do not consider it fruitful to attempt to make a general classification of those infinite dimensional problems to which our results can be applied, it is apparent that our methods of proof, namely either homotopy or analytic perturbation theory, are not intrinsically finite dimensional. Accordingly our analysis could be extended to provide analogous theorems pertaining to various classes of Hamiltonian systems of partial differential equations. A rigorous analysis of such examples is necessarily more technical, and we shall not pursue such a course here.

Perturbation analysis of Hamiltonian systems received much attention in the 1950's, primarily in the context of perturbations that preserve the Hamiltonian structure and that therefore also preserve the two-fold symmetry of the eigenvalues [8], [12], [19]. Surveys may be found in [7], [9], [14], [24]. In contrast to these works, our main interest is in the effect of dissipative perturbations that break the Hamiltonian structure. Effects of dissipation on the dynamics of mechanical systems were extensively considered by Rayleigh and Kelvin amongst others [5] Article 40, [13] Chpt. XI, [10], [17] Chpt. 6.10, [20] Chpt. X, [23] Article 345, [25] Chpt. 4.2. Various connections between the two eigenvalue problems (1.4) and (1.5) were known to these authors. For example, if (1.5) has an odd number of negative eigenvalues, then (1.4) has at least one real pair of eigenvalues. Alternatively for canonical systems where (1.2) holds and the Hamiltonian is of a special decoupled form involving the sum of a positive definite quadratic kinetic energy and a potential, there is an immediate sharp correspondence between the locations of eigenvalues of (1.5) and of (1.4). It was also realised that in the absence of decoupling there was no such simple relation, and the coupling terms are often referred to as *gyroscopic* because they are responsible for the stabilization of an upright spinning top.

Dissipative perturbations and eigenvalue problems arising in Hamiltonian systems are still of contemporary interest, c.f. [15] and [4]. The results obtained in these two works are described in more detail in, respectively, Sections 8 and 9, but the primary contrasts are that our analysis permits *arbitrary* Jordan structure in the unperturbed eigenvalue problem (1.4), and obtains a *precise* count on the number of perturbed eigenvalues in the right half-plane.

Our presentation is organized as follows. The correspondence between the eigenvalues of (1.5) and (1.7) is stated and proven in Section 2 using a homotopy argument. A general perturbation result is also stated in Section 2, but the proofs and detailed analysis of eigenvalue behavior are deferred to Section 4. Section 3 treats the special case where any multiple eigenvalue is semisimple, i.e. has a complete set of eigenvectors. The notation required in the general perturbation analysis is rather complicated, so in Section 5 we illustrate the theory through a comparatively straightforward example involving a non-semisimple eigenvalue. Sections 6 and 7 generalize our analysis to cases where the perturbation D is, respectively, nonsymmetric but still semidefinite, and skew-symmetric. Connections with classic results for Hamiltonian perturbation theory are described in Section 8. Section 9 demonstrates that our results apply to many standard mechanical systems with second-order Lagrangian time dynamics perturbed by Rayleigh dissipation.

2. Main Results

Assume J and S are real, square, nonsingular $2n \times 2n$ matrices, with J skew-symmetric ($J = -J^T$) and S symmetric ($S = S^T$). Consider dissipative perturbations of the form

$$A(\epsilon) = (J - \epsilon D)S \tag{2.1}$$

where $\epsilon > 0$ is a real parameter and D is a real, symmetric, positive semidefinite matrix. Let z^* denote the complex conjugate transpose of z.

THEOREM 1. *Suppose that*

(2.2) $z^* SDSz > 0, \quad \forall$ eigenvectors z of JS with pure imaginary eigenvalues.

Then, for all $\epsilon > 0$ and counting algebraic multiplicity, the number of eigenvalues of $A(\epsilon)$ in the right half-plane equals the number of negative eigenvalues of S. Furthermore, no eigenvalue of $A(\epsilon)$ is imaginary for $\epsilon > 0$.

Proof: Consider the family of eigenvalue problems

$$(\tau(J - \epsilon D) - (1-\tau)I)\, Su = \mu u, \qquad 0 \le \tau \le 1. \tag{2.3}$$

We claim that (2.3) has no eigenvalue on the imaginary axis for any value of $\tau \in [0,1]$. Suppose the contrary, i.e. (2.3) holds with μ pure imaginary and u nonzero. Take the scalar product of Su with (2.3) and rearrange to obtain

$$\tau u^* SJSu - u^* S\,(\tau\epsilon D + (1-\tau)I)\, Su = \mu u^* Su. \tag{2.4}$$

Using the symmetry of S and D, the skew-symmetry of J, and the fact that ϵ is real and μ is imaginary, we find, upon taking the real part of (2.4), that

$$u^* S\,(\tau\epsilon D + (1-\tau)I)\, Su = 0.$$

This identity contradicts the properties of D and S. In particular, we have $\epsilon > 0$, D positive semidefinite and S nonsingular, so for $0 \le \tau < 1$, we must have $u = 0$, and the contradiction is immediate. For $\tau = 1$ we still obtain an immediate contradiction unless $DSu = 0$, in which case u is also an eigenvector of JS with imaginary eigenvalue μ, which contradicts (2.2).

It may therefore be concluded that (2.3) has no eigenvalue on the imaginary axis for any $\epsilon > 0$ and $0 \le \tau \le 1$. It follows from continuous dependence of eigenvalues on the parameter τ that the number of eigenvalues (counting algebraic multiplicity) in each half-plane is a homotopy invariant, i.e. is constant for all $0 \le \tau \le 1$. Theorem 1 follows, since for $\tau = 1$ the matrix in (2.3) reduces to (1.4), while for $\tau = 0$ it reduces to $-S$. Q.E.D.

The homotopy is also applicable in the case $\epsilon = 0$, where it allows us to conclude that for $0 \le \tau < 1$, (2.3) has no eigenvalue on the imaginary axis. However for $\epsilon = 0$, as $\tau \to 1$ eigenvalues can approach the imaginary axis. Now (2.3) involves only real matrices, so eigenvalues must occur in complex conjugate pairs. It follows that if S has an odd number of negative eigenvalues, at least one eigenvalue in the right half-plane remains real as $\tau \to 1$. By assumption JS is nonsingular, so this eigenvalue cannot converge to zero. Thus at critical points of odd index, JS has at least one positive real eigenvalue. This conclusion is the classic result already mentioned in the introduction ([23] Article 345^x, [5] Article 40). In the finite dimensional case a more elementary proof is available by simply considering the determinant of JS. However, the more constructive homotopy proof focuses attention on the main difficulty, namely movement of pure imaginary eigenvalues of JS under perturbation.

For ϵ small, and for a given imaginary eigenvalue λ, Theorem 2 below gives more detailed information describing the perturbed eigenvalues associated with λ. Before stating Theorem 2 we need to introduce some further notation.

Any eigenvalue of JS that is not pure imaginary is of little interest, since small perturbations cannot move it across the imaginary axis. Accordingly denote the *imaginary* eigenvalues of JS by

$$\lambda_1, \ldots, \lambda_\nu, -\lambda_1, \ldots, -\lambda_\nu,$$

where λ_j and $-\lambda_j$ have algebraic multiplicity m_j, and λ_j is positive imaginary. Denote the associated eigenvalues of $A(\epsilon)$ by the continuous functions

$$\mu_j^q(\epsilon), \ \overline{\mu_j^q(\epsilon)}$$

with the understanding that

$$\mu_j^q(0) = \lambda_j, \qquad q = 1, \ldots, m_j, \quad j = 1, \ldots, \nu.$$

Recall that the *geometric eigenspace* for λ_j is the null space of $JS - \lambda_j I$, and the *invariant subspace* for λ_j is the null space of $(JS - \lambda_j I)^{m_j}$.

THEOREM 2. *Let $j \in \{1, \ldots, \nu\}$, and assume that*

SDS restricted to the geometric eigenspace for λ_j

is positive definite. Then, for ϵ sufficiently small, the number of the eigenvalues $\mu_j^q(\epsilon)$ in the right half-plane is equal to the number of negative eigenvalues of

S restricted to the invariant subspace for λ_j.

If λ_j is simple, Theorem 2 follows easily from well known perturbation results. But if λ_j has multiplicity $m_j > 1$, the behavior of the eigenvalues of the perturbed matrix is potentially quite complicated. We emphasize that both Theorems 1 and 2 hold regardless of the Jordan structure of JS.

Recall that an eigenvalue λ_j is *semisimple* if the geometric eigenspace and the invariant subspace corresponding to λ_j are equal; equivalently, the part of the Jordan form of JS corresponding to λ_j is diagonal. An eigenvalue λ_j is *nonderogatory* if the geometric eigenspace for λ_j is a one-dimensional subspace of the invariant subspace for λ_j; equivalently, the part of the Jordan form of JS corresponding to λ_j consists of a single Jordan block. In the analysis of multiple eigenvalues, semisimple and nonderogatory eigenvalues lie at opposite structural extremes (minimum and maximum number of entries in the corresponding off-diagonal positions of the Jordan form). A simple eigenvalue is both nonderogatory and semisimple.

The proof of Theorem 2 is deferred to Section 4. There we give a detailed analysis of the relationship between the inertia of S and the asymptotic behavior of the eigenvalues of $A(\epsilon)$ in the case of general Jordan structure. This result is expressed as Theorem 4. The proof of Theorem 2 demonstrates that hypothesis (2.2) is sharp in the sense that Theorems 1 and 2 are false when (2.2) does not hold.

Since the case in which λ_j is semisimple is considerably more straightforward, we address it first, summarizing the asymptotic behavior of the perturbed eigenvalues in that case in Theorem 3.

3. Perturbation Formulas in the Semisimple Case

THEOREM 3. *Let $j \in \{1, \dots, \nu\}$, and assume that the imaginary eigenvalue λ_j is semisimple. Let Z_j be a matrix whose m_j columns form a basis for the geometric eigenspace (equivalently the invariant subspace) for λ_j. Without loss of generality, Z_j can be chosen such that*

$$Z_j^* S Z_j = K_j = \mathrm{Diag}(\pm 1). \tag{3.1}$$

Let $T_j = Z_j^* SDSZ_j$. Then

$$\mu_j^q(\epsilon) = \lambda_j + \xi_j^q \epsilon + o(\epsilon), \quad q = 1, \dots, m_j, \tag{3.2}$$

where the ξ_j^q, $q = 1, \dots, m_j$, are the eigenvalues of the matrix $-K_j T_j$.

Proof: That (3.1) holds without loss of generality follows from choosing an arbitrary eigenvector basis Z_j, diagonalizing $Z_j^* S Z_j$, and scaling. The number of ± 1's depends only on the inertia of S restricted to the geometric eigenspace. We have $JSZ_j = \lambda_j Z_j$ so

$$\begin{aligned} Z_j^* SJS &= -(JSZ_j)^* S \\ &= -(\lambda_j Z_j)^* S \\ &= \lambda_j Z_j^* S. \end{aligned}$$

Thus, the rows of $Z_j^* S$ form a basis for the *left* geometric eigenspace of JS. Noting that (3.1) holds, we see that choosing

$$Y_j = K_j Z_j^* S$$

yields bases Z_j and Y_j for the right and left eigenspaces that satisfy the biorthogonality condition

$$Y_j Z_j = I.$$

Therefore, $\Pi_j = Z_j Y_j$ is the eigenprojection for λ_j, satisfying

$$\Pi_j JS = JS\Pi_j = \Pi_j JS \Pi_j = \lambda_j \Pi_j.$$

Observe that

$$\frac{d}{d\epsilon} A(0) = -DS. \tag{3.3}$$

Applying the well known perturbation theory for semisimple eigenvalues [11], Theorem II.5.4, we see that (3.2) holds, where the ξ_j^q are the nontrivial eigenvalues of $\Pi_j DS \Pi_j$, i.e. the eigenvalues of the $m_j \times m_j$ matrix

$$Y_j(-DS)Z_j = -K_j Z_j^* SDSZ_j = -K_j T_j. \qquad \text{Q.E.D.} \tag{3.4}$$

Now assume that SDS is positive definite on the geometric eigenspace for λ_j, i.e. the Hermitian matrix T_j is positive definite. Thus the Hermitian square root $T_j^{1/2}$ exists, with $K_j T_j$ similar to $T_j^{1/2} K_j T_j^{1/2}$. It follows that the ξ_j^q are all real and, by Sylvester's law of inertia, their signs are determined by the sign pattern of the matrix K_j. Consequently, Theorem 2 follows from Theorem 3 in the semisimple case.

The signs in $K_j = Z_j^* S Z_j$ are called the *Krein signatures* associated with the positive imaginary eigenvalue λ_j. This nomenclature is discussed further in Section 8.

Theorem 3 states that the perturbed eigenvalues corresponding to the semisimple eigenvalue λ_j are differentiable with respect to the perturbation parameter ϵ, with derivatives equal to the eigenvalues of the $m_j \times m_j$ matrix $K_j T_j$. The matrix $K_j T_j$, like the unperturbed matrix JS, is the product of two matrices, neither of which is positive definite in general. In the introduction, it was noted that if S is definite, the eigenvalues of JS are imaginary; likewise the proof of Theorem 2 in the semidefinite case uses the hypothesis that T_j is definite, so that the eigenvalues of the product are real with signs determined by the signs of the factor K_j. We shall use this kind of argument repeatedly, although it becomes more complicated in the case of general Jordan structure.

If λ_j is simple, $K_j T_j$ is just a scalar. Let z_j denote the corresponding eigenvector. In this case, Theorem 3 reduces to the statement that the corresponding perturbed eigenvalue is

$$\mu_j(\epsilon) = \lambda_j + \xi_j \epsilon + o(\epsilon) \tag{3.5}$$

where

$$\xi_j = -\kappa_j \tau_j, \quad \tau_j = z_j^* SDS z_j, \quad \kappa_j = z_j^* S z_j = \pm 1.$$

It should be noted that since we are considering only real perturbations, there is no need to consider conjugate eigenvalues separately, since movement of eigenvalues in the upper half-plane determines the movement of those in the lower half-plane.

4. Perturbation Formulas in the General Case

In the case of arbitrary Jordan structure, the behavior of the eigenvalues under perturbation is quite complicated. We shall need two powerful branches of matrix and operator theory: the theory of indefinite scalar products [9], and analytic perturbation theory for eigenvalues [2]. We begin by establishing notation. We shall assume in the following that JS has no eigenvalue in addition to the nonzero imaginary numbers $\pm\lambda_1, \dots, \pm\lambda_\nu$. Extension to the more general case is straightforward, but notationally inconvenient, and is deferred to the end of this section.

Let V reduce JS to Jordan form, i.e.

$$V^{-1}JSV = \mathrm{Diag}(G_1, \dots, G_\nu, \overline{G_1}, \dots, \overline{G_\nu}), \tag{4.1}$$

with

$$G_j = \mathrm{Diag}(\underbrace{\Gamma_j^1, \dots, \Gamma_j^1}_{r_j^1}, \dots\dots, \underbrace{\Gamma_j^{p_j}, \dots, \Gamma_j^{p_j}}_{r_j^{p_j}}), \tag{4.2}$$

and

$$\Gamma_j^k = \begin{bmatrix} \lambda_j & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & \lambda_j \end{bmatrix}$$

a Jordan block for λ_j with dimension m_j^k. The Jordan block Γ_j^k appears r_j^k times in G_j, the part of the Jordan form corresponding to λ_j. The m_j^k, $k = 1, \dots, p_j$, are called the *distinct partial multiplicities* for λ_j, and they satisfy

$$m_j^1 > \cdots > m_j^{p_j}, \qquad \sum_{k=1}^{p_j} r_j^k m_j^k = m_j.$$

If λ_j is semisimple, then $p_j = 1$, $m_j^1 = 1$ and $r_j^1 = m_j$. If λ_j is nonderogatory, then $p_j = 1$, $m_j^1 = m_j$, and $r_j^1 = 1$. The columns of V are called Jordan vectors (alternatively principal vectors or generalized eigenvectors). We write

$$V = [V_1, \dots, V_\nu, \overline{V_1}, \dots, \overline{V_\nu}]$$

conformally with (4.1) and

$$V_j = [V_j^{11}, \ldots, V_j^{1r_j^1}, \ldots\ldots, V_j^{p_j 1}, \ldots, V_j^{p_j r_j^{p_j}}] \tag{4.3}$$

conformally with (4.2). Each V_j^{kl} represents a distinct chain, of length m_j^k, of Jordan vectors for λ_j, since

$$(JS - \lambda_j)V_j^{kl} = V_j^{kl} \begin{bmatrix} 0 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & 0 \end{bmatrix}. \tag{4.4}$$

LEMMA 1. *The matrix V in (4.1) can be chosen so that*

$$V^* S V = \mathrm{Diag}(H_1, \ldots, H_\nu, \overline{H_1}, \ldots, \overline{H_\nu}) \tag{4.5}$$

where

$$H_j = \mathrm{Diag}(\kappa_j^{11}\Sigma_j^1, \ldots, \kappa_j^{1r_j^1}\Sigma_j^1, \ldots\ldots, \kappa_j^{p_j 1}\Sigma_j^{p_j}, \ldots, \kappa_j^{p_j r_j^{p_j}}\Sigma_j^{p_j})$$

conformally with (4.1),(4.2). Here

$$\kappa_j^{kl} = \pm 1$$

and

$$\Sigma_j^k = \begin{bmatrix} & & & & \sigma_j^k \\ & & & -\sigma_j^k & \\ & & \sigma_j^k & & \\ & \cdot & & & \\ \cdot & & & & \end{bmatrix}, \tag{4.6}$$

a reverse diagonal matrix of dimension m_j^k, with

$$\sigma_j^k = \left\{ \begin{array}{ll} 1 & m_j^k \equiv 1 \pmod 4 \\ i & m_j^k \equiv 2 \pmod 4 \\ -1 & m_j^k \equiv 3 \pmod 4 \\ -i & m_j^k \equiv 0 \pmod 4 \end{array} \right\}, \quad i = \sqrt{-1}. \tag{4.7}$$

Proof: Consider the matrix iJS. We have

$$(iJS)^* S = S(iJS)$$

so iJS is said to be self-adjoint with respect to the indefinite scalar product defined by $\langle x, x\rangle = x^*Sx$ [9], Section I.2.2. The eigenvalues of iJS and those of JS differ by factors of i, so, since the latter are imaginary, the former are real. The Jordan form of iJS has a structure identical to the Jordan form of JS, given by (4.1). Let $\widetilde{V}$ denote a similarity transformation reducing iJS to Jordan form, with blocks $\widetilde{V}_j^{kl}$ corresponding to the blocks V_j^{kl} of (4.3), and let $\widetilde{\Gamma_j^k}$ denote the associated Jordan blocks. The diagonals of Γ_j^k and $\widetilde{\Gamma_j^k}$ differ by a factor of i, but the superdiagonals are the same. We have

$$i\Gamma_j^k = \Omega^*\widetilde{\Gamma_j^k}\Omega \quad \text{where} \quad \Omega = \mathrm{Diag}(1, i, -1, -i, \ldots).$$

Therefore, the $\widetilde{V}_j^{kl}$ and V_j^{kl} are conveniently related by

$$V_j^{kl} = \widetilde{V}_j^{kl}\Omega. \tag{4.8}$$

By [9], Section I.3.2, $\widetilde{V}$ can be chosen so that

$$\widetilde{V}^*S\widetilde{V} = \mathrm{Diag}(\widetilde{H}_1, \ldots, \widetilde{H}_\nu, \widetilde{H}_1', \ldots, \widetilde{H}_\nu') \tag{4.9}$$

with

$$\widetilde{H}_j = \mathrm{Diag}(\kappa_j^{11}P_j^1, \ldots, \kappa_j^{1r_j^1}P_j^1, \ldots\ldots, \kappa_j^{p_j1}P_j^{p_j}, \ldots, \kappa_j^{p_jr_j^{p_j}}P_j^{p_j})$$

and

$$\widetilde{H}_j' = \mathrm{Diag}((\kappa_j^{11})'P_j^1, \ldots, (\kappa_j^{1r_j^1})'P_j^1, \ldots\ldots, (\kappa_j^{p_j1})'P_j^{p_j}, \ldots, (\kappa_j^{p_jr_j^{p_j}})'P_j^{p_j})$$

where

$$\kappa_j^{kl} = \pm 1, \qquad (\kappa_j^{kl})' = \pm 1$$

and

$$P_j^k = \begin{bmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{bmatrix},$$

a reversing permutation matrix of dimension m_j^k. Using (4.8) and (4.9), we see that V^*SV is also block diagonal, with

$$(V_j^{kl})^*SV_j^{kl} = \Omega^*(\widetilde{V}_j^{kl})^*S\widetilde{V}_j^{kl}\Omega = \kappa_j^{kl}\Omega^*P_j^k\Omega = \kappa_j^{kl}\Sigma_j^k. \tag{4.10}$$

This proves Lemma 1. Note that applying the same argument to the conjugate blocks in the second half of V shows that

$$(\overline{V_j^{kl}})^*S\overline{V_j^{kl}} = (\kappa_j^{kl})'\Sigma_j^k, \tag{4.11}$$

so that, since identities (4.10) and (4.11) are conjugates of each other,

$$(\kappa_j^{kl})' = \left\{ \begin{array}{ll} \kappa_j^{kl} & m_j^k \text{ is odd} \\ -\kappa_j^{kl} & m_j^k \text{ is even} \end{array} \right\}. \qquad \text{Q.E.D.}$$

Lemma 1 displays the inertia of S in the convenient form 4.5. The backwards diagonal matrix Σ_j^k is Hermitian and unitary with half of its eigenvalues equal to $+1$ and half of them equal to -1. When m_j^k is odd the extra eigenvalue is $+1$, since the trace of Σ_j^k is one. The presence of Jordan blocks in the Jordan form of JS therefore places strong restrictions on the inertia of S restricted to the corresponding invariant subspace. Specifically, if JS has a Jordan block of size m, where m is even, then S, restricted to the corresponding invariant subspace, must have $m/2$ positive and $m/2$ negative eigenvalues, and the same is true on the invariant subspace for the conjugate eigenvalue. This fact is at the heart of the theory of indefinite scalar products [9]. It generalizes the well known fact that the presence of any nontrivial Jordan block in the spectral decomposition of a matrix precludes the possibility that the matrix is self-adjoint with respect to the usual definite inner product.

We are now ready to state and prove the main result of this section. Assume that V satisfies the conditions expressed in Lemma 1. Let z_j^{kl} denote the first column of V_j^{kl}, $l = 1, \ldots, r_j^k$, $k = 1, \ldots, p_j$, $j = 1, \ldots, \nu$, and let

$$Z_j^k = [z_j^{k1}, \ldots, z_j^{kr_j^k}].$$

The columns of Z_j^k form a set of r_j^k eigenvectors of grade m_j^k for λ_j, i.e. they are associated with r_j^k distinct Jordan chains of length m_j^k. Let

$$Z_j = [Z_j^1, \ldots, Z_j^{p_j}].$$

The columns of Z_j span the geometric eigenspace for λ_j. Also, let

$$K_j^k = \mathrm{Diag}(\kappa_j^{k1}, \ldots, \kappa_j^{kr_j^k}), \tag{4.12}$$

denote the diagonal matrix of signs.

THEOREM 4. *Let $j \in \{1, \ldots, \nu\}$, and assume that SDS is nonsingular on the geometric eigenspace for the imaginary eigenvalue λ_j, i.e. $Z_j^* SDSZ_j$ is nonsingular. Then the eigenvalues of $A(\epsilon)$ converging to λ_j as $\epsilon \to 0$ consist of groups*

$$\mu_j^{klq}(\epsilon) = \lambda_j + (\xi_j^{kl})^{1/m_j^k} \epsilon^{1/m_j^k} + o(\epsilon^{1/m_j^k}) \tag{4.13}$$

$$q = 1, \ldots, m_j^k, \;\; l = 1, \ldots, r_j^k, \;\; k = 1, \ldots, p_j$$

where

(i) the notation $\mu_j^q(\epsilon)$ *used in Theorem 2 has been refined to the more specific notation* $\mu_j^{klq}(\epsilon)$ *which groups the eigenvalues of* $A(\epsilon)$ *according to the Jordan structure of* JS

(ii) the equation (4.13) means that the distinct values $\mu_j^{klq}(\epsilon)$*, for* $q = 1, \ldots, m_j^k$*, are defined by taking the* m_j^k *distinct* m_j^k*-th roots of* ξ_j^{kl}

(iii) for $k = 1$*, i.e. the case corresponding to Jordan blocks in* G_j *with largest dimension, the* ξ_j^{kl}*, that is* ξ_j^{1l}*,* $l = 1, \ldots, r_j^1$*, are the eigenvalues of the* r_j^1 *by* r_j^1 *matrix*

$$-\overline{\sigma_j^1} K_j^1 T_j^1,$$

where

$$T_j^1 = (Z_j^1)^* SDSZ_j^1 \tag{4.14}$$

(iv) for $k > 1$*, the* ξ_j^{kl}*,* $l = 1, \ldots, r_j^k$*, are the eigenvalues of the* r_j^k *by* r_j^k *matrix*

$$-\overline{\sigma_j^k} K_j^k T_j^k, \tag{4.15}$$

where, omitting sub- and superscripts for brevity

$$T_j^k = C - B^* F^{-1} B,$$

with

$$\begin{bmatrix} F & B \\ B^* & C \end{bmatrix} = \begin{bmatrix} (X_j^{k-1})^* \\ (Z_j^k)^* \end{bmatrix} SDS \, [\, X_j^{k-1} \quad Z_j^k \,] \tag{4.16}$$

and

$$X_j^{k-1} = [Z_j^1, \ldots, Z_j^{k-1}].$$

Proof: Equation (4.4) shows that the columns of V_j^{kl} form a chain, of length m_j^k, of Jordan vectors for λ_j. Furthermore,

$$\begin{aligned} (V_j^{kl})^* S(JS - \lambda_j) &= ((\lambda_j - JS)V_j^{kl})^* S \\ &= \begin{bmatrix} 0 & & & & \\ -1 \cdot & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & -1 & 0 \end{bmatrix} (V_j^{kl})^* S \end{aligned}$$

i.e. the *rows* of $(V_j^{kl})^* S$ form a chain, of length m_j^k, of *left* Jordan vectors for λ_j. Reversing the order of this chain and scaling appropriately gives

$$U_j^{kl} = \kappa_j^{kl} \Sigma_j^k (V_j^{kl})^* S,$$

which satisfies

$$U_j^{kl}(JS - \lambda_j) = \begin{bmatrix} 0 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & 0 \end{bmatrix} U_j^{kl}.$$

Furthermore, from (4.5), we have

$$U_j^{kl} V_j^{kl} = \kappa_j^{kl} \Sigma_j^k (V_j^{kl})^* S V_j^{kl} = (\kappa_j^{kl})^2 (\Sigma_j^k)^2 = I$$

a biorthogonality condition on the right chain defined by the columns of V_j^{kl} and the left chain defined by the rows of U_j^{kl}. Indeed, collecting all left chains U_j^{kl} in a matrix U, we have the biorthogonality condition $UV = I$. For each pair of chains V_j^{kl}, U_j^{kl}, the associated right and left eigenvectors are respectively the *first* column of V_j^{kl}, denoted z_j^{kl}, and the *last* row of U_j^{kl}, namely

$$y_j^{kl} = \kappa_j^{kl} \overline{\sigma_j^k} (z_j^{kl})^* S.$$

Collect all such left eigenvectors for λ_j associated with chains of length m_j^k in a matrix

$$Y_j^k = [y_j^{k1}, \ldots, y_j^{kr_j^k}] = \overline{\sigma_j^k} K_j^k (Z_j^k)^* S, \tag{4.17}$$

recalling that K_j^k was defined in (4.12).

We are now in a position to apply the analytic perturbation theory for multiple eigenvalues with arbitrary Jordan structure, which was developed by Lidskii and others in the Russian literature in the 1960's and refined and extended by Baumgärtel [2]. Specifically, we shall use [2] Thm. 7.4.7 (p. 306) together with Thm. 7.4.6 (p. 305) and Lemma 7.4.9 (p. 298). Using (3.3), these show that the eigenvalues of $A(\epsilon)$ associated with the largest Jordan blocks for λ_j, namely those of dimension m_j^1, have expansions given by

$$\mu_j^{1lq}(\epsilon) = \lambda_j + (\xi_j^{1l})^{1/m_j^1} \epsilon^{1/m_j^1} + o(\epsilon^{1/m_j^1}),$$

where the ξ_j^{1l} are the eigenvalues of the r_j^1 by r_j^1 matrix

$$Y_j^1(-DS)Z_j^1 = -\overline{\sigma_j^1} K_j^1 (Z_j^1)^* SDSZ_j^1 = -\overline{\sigma_j^1} K_j^1 T_j^1.$$

This proves the theorem for the case $k = 1$. The case $k > 1$ is more complicated. Using

$$X_j^{k-1} = [Z_j^1, \ldots, Z_j^{k-1}] \quad \text{and} \quad W_j^{k-1} = \begin{bmatrix} Y_j^1 \\ \vdots \\ Y_j^{k-1} \end{bmatrix},$$

the results in Baumgärtel just quoted show that, provided $W_j^{k-1}DSX_j^{k-1}$ is nonsingular, the eigenvalues of $A(\epsilon)$ associated with Jordan blocks of size m_j^k for λ_j have expansions given by (4.13), where the ξ_j^{kl} are the roots of the polynomial in ξ

$$\det\left(\begin{bmatrix} W_j^{k-1} \\ Y_j^k \end{bmatrix}(-DS)\,[\,X_j^{k-1} \quad Z_j^k\,] \quad - \quad \xi \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\right), \tag{4.18}$$

the identity block having dimension r_j^k. Let

$$E_j^{k-1} = \mathrm{Diag}(\overline{\sigma_j^1}K_j^1, \ldots, \overline{\sigma_j^{k-1}}K_j^{k-1}),$$

and observe, using (4.17), that $W_j^{k-1}DSX_j^{k-1} = E_j^{k-1}F$, where F is defined in (4.16). Thus, the assumption that SDS is nonsingular on the geometric eigenspace provides the required nonsingularity condition. Furthermore, (4.18) reduces, using (4.16) and (4.17), to

$$-\det\left(\begin{bmatrix} E_j^{k-1} & 0 \\ 0 & \overline{\sigma_j^k}K_j^k \end{bmatrix}\begin{bmatrix} F & B \\ B^* & C+\xi I \end{bmatrix}\right). \tag{4.19}$$

We have

$$\begin{aligned} \det\begin{bmatrix} F & B \\ B^* & C+\xi I \end{bmatrix} &= \det\left(\begin{bmatrix} I & 0 \\ -B^*F^{-1} & I \end{bmatrix}\begin{bmatrix} F & B \\ B^* & C+\xi I \end{bmatrix}\right) \\ &= \det\begin{bmatrix} F & B \\ 0 & C-B^*F^{-1}B+\xi I \end{bmatrix} \end{aligned}$$

so the roots of (4.19) are the eigenvalues of (4.15), which completes the proof of Theorem 4. Q.E.D.

In the semisimple case, Theorem 4 reduces to Theorem 3 since all Jordan blocks have dimension one, all Jordan vectors are eigenvectors and $\overline{\sigma_j^1}K_j^1T_j^1$ reduces to K_jT_j of Theorem 3. In the nonderogatory case, λ_j is again associated with only one size Jordan block, namely a single block of size m_j. In this case $\overline{\sigma_j^1}K_j^1T_j^1$ is a scalar, and the associated perturbed eigenvalues split along rays separated by angles of $2\pi/m_j$. In the case that λ_j is simple, both these special cases reduce to (3.5).

We are now ready to prove Theorem 2, stated in Section 2.

Proof of Theorem 2: Consider a group of eigenvalues $\mu_j^{klq}(\epsilon)$ corresponding to Jordan blocks for the imaginary eigenvalue λ_j with size m_j^k. By assumption, SDS is positive definite on the geometric eigenspace. This ensures, in the case $k=1$, that T_j^1 is positive definite, and, in the case $k>1$, that (4.16) is positive definite and therefore that the Schur complement $T_j^k = C - B^*F^{-1}B$ is positive definite. Consequently, using (4.7), the eigenvalues ξ_j^{kl} of (4.15) are real if m_j^k is odd and imaginary if m_j^k is even, with sign pattern determined by $-\overline{\sigma_j^k} K_j^k = -\overline{\sigma_j^k}\mathrm{Diag}(\kappa_j^{kl})$. The m_j^k m_j^k-th roots of each ξ_j^{kl} are equally spaced around a circle centered at the origin in the complex plane. In the case that m_j^k is even, none of the m_j^k m_j^k-th roots of the imaginary quantities ξ_j^{kl} are either real or imaginary, so, for each eigenvalue ξ_j^{kl}, half of the m_j^k-th roots lie strictly in the left half-plane and half in the right. Now suppose that m_j^k is odd, so that the ξ_j^{kl} are real with signs determined by $\alpha_j^{kl} = -\overline{\sigma_j^k}\kappa_j^{kl}$, $l = 1, \ldots, r_j^k$. The only real m_j^k-th root of α_j^{kl} has the same sign as α_j^{kl}. One must then distinguish between the cases $m_j^k \equiv 1 \pmod 4$ and $m_j^k \equiv 3 \pmod 4$ to count how many roots lie in the left half-plane and how many in the right. It is easily seen that, in both cases, $(m_j^k + \kappa_j^{kl})/2$ of the m_j^k-th roots of ξ_j^{kl} lie strictly in the left half-plane, and the others lie in the right. It follows from (4.13) that for ϵ sufficiently small, the half-plane in which $\mu_j^{klq}(\epsilon)$ lies is completely determined by the half-plane in which the corresponding m_j^k-th root of ξ_j^{kl} lies.

To complete the proof of Theorem 2, we need only determine the inertia of S restricted to the invariant subspace for λ_j. This is displayed in the block diagonal form (4.5). Specifically, we have, for $l = 1, \ldots, r_j^k$, $k = 1, \ldots, p_j$

$$(V_j^{kl})^* S V_j^{kl} = \kappa_j^{kl}\Sigma_j^k. \tag{4.20}$$

In the case m_j^k is even, half of the eigenvalues of (4.20) are positive and half are negative, while if m_j^k is odd, exactly $(m_j^k + \kappa_j^{kl})/2$ are positive and the rest are negative. Thus, for ϵ sufficiently small, the number of eigenvalues $\mu_j^{klq}(\epsilon)$, $q = 1, \ldots, m_j^k$, in the right half-plane and the number of positive eigenvalues of (4.20) is the same. Since (4.5) is block diagonal, the number of negative eigenvalues of S restricted to the invariant subspace is obtained from summing the number of negative eigenvalues in the components 4.20. This proves Theorem 2. Q.E.D.

A restricted version of Theorem 1, in which ϵ is assumed sufficiently small, may now be proved using Theorem 2 together with Lemma 1. Before we can do this, we must remove the assumption that all eigenvalues of JS are imaginary, as follows. By two-fold symmetry, all non-imaginary eigenvalues occur in pairs $(\lambda, -\overline{\lambda})$. These eigenvalues cannot cross the imaginary axis under infinitesimal perturbation, so there is no need to analyze their behavior under perturbation: the number in the left half-plane is the same as the number in the right half-plane. We now need to verify that the counts of positive and negative eigenvalues of S restricted to the corresponding invariant subspaces are the same. In order to do this, we must extend the block diagonal decomposition (4.5) to cover non-imaginary eigenvalues. The pair $(\lambda, -\overline{\lambda})$ for JS corresponds to a complex conjugate pair $(i\lambda, -i\overline{\lambda})$ for iJS. Using [9], Sec I.3.2, we see that each such pair, with corresponding Jordan vectors in V, introduces an additional set of matrices of the form Σ_j^k, into (4.5). The number and dimensions of these blocks depend on the Jordan form of the pair $(\lambda, -\overline{\lambda})$, but since the dimensions are all even, the contributions to the number of positive and negative eigenvalues of S restricted to the corresponding invariant subspace are the same.

The proof that Theorem 1 holds for sufficiently small ϵ is then immediate. The inertia of S and of V^*SV are the same, and the latter is displayed in the block diagonal form (4.5). Therefore, the number of negative eigenvalues of S is obtained by adding together the number of negative eigenvalues of S restricted to each of the invariant subspaces. The movement of the conjugate eigenvalues in the lower half-plane is determined by the eigenvalues in the upper half-plane.

Finally, we note that Theorems 2, 3 and 4 and the restricted version of Theorem 1 (for ϵ sufficiently small) all hold when the linear perturbation $(J - \epsilon D)S$ is replaced by an analytic perturbation $(J - \epsilon D - \epsilon^2 D' - \cdots)S$. The proofs are unchanged.

5. Example

In order to illustrate the rather intricate arguments required in the statement and proof of Theorem 4, we shall now describe a comparatively straightforward example in which all computations can be carried out explicitly, and the eigenvalues of the perturbed problem can easily be determined numerically. We consider the simplest case where a multiple eigenvalue is neither semisimple nor nonderogatory. Let $n = 3$, so that JS has order six, and suppose JS has one conjugate pair of imaginary eigenvalues $\pm\lambda_1$, each of multiplicity three. Suppose further that λ_1 has two independent eigenvectors, i.e. $L = JS - \lambda_1 I$ has nullity two. It follows that λ_1 has two Jordan blocks, of sizes two and one. Using standard methods it is easy to compute vectors a, b and c satisfying $Lb = a$, $La = 0$ and $Lc = 0$. Thus a and b form a Jordan chain of length two and c is a Jordan chain of length one. Both a and c are eigenvectors, but b is a Jordan vector, or generalized eigenvector. Furthermore, a^*S and $-b^*S$ form a *left* Jordan chain of length two, and c^*S a left Jordan chain of length one.

Multiplying the equation $Lb = a$ on the left by the left eigenvector a^*S, we see that $a^*Sa = 0$, and doing the same with the left eigenvector c^*S yields $c^*Sa = 0$. Similarly multiplying the same equation on the left by the left Jordan vector $-b^*S$, we find that $a^*Sb = -b^*Sa$. It follows that a^*Sb is imaginary. We therefore have

$$[\,a \quad b \quad c\,]^* S [\,a \quad b \quad c\,] = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & \beta & \gamma \\ 0 & \overline{\gamma} & \delta \end{bmatrix} \tag{5.1}$$

where α is imaginary. Furthermore, it is easy to show from the assumption on the Jordan form that α and δ are both nonzero.

We now wish to normalize a, b and c such that (5.1) is consistent with the block diagonal form given in Lemma 1. First scale a and b by $1/\sqrt{|a^*Sb|}$, so that $\alpha = a^*Sb = \pm i$. Then replace b by $b - \alpha(b^*Sb)/2\,a$, so that $b^*Sb = 0$. These rescalings do not change the property that a and b form a Jordan chain, and do not affect the value of a^*Sb since $a^*Sa = 0$. Then replace c by the eigenvector $c - \alpha(b^*Sc)a$, so that $b^*Sc = 0$. Finally, scale the new choice of c by $1/\sqrt{|c^*Sc|}$, so that $c^*Sc = \pm 1$. Then (5.1) holds with $\alpha = \pm i$, $\beta = \gamma = 0$, and $\delta = \pm 1$. Thus with $V_1 = [a\ b\ c]$, $V = [V_1\ \overline{V_1}]$ satisfies (4.5).

In the notation of Section 4 we have $m_1 = 3$, $p_1 = 2$, $m_1^1 = 2$, $r_1^1 = 1$, $m_1^2 = 1$, $r_1^2 = 1$, $\sigma_1^1 = i$, and $\sigma_1^2 = 1$. Furthermore $V_1 = [V_1^{11}\ V_1^{21}]$, where $V_1^{11} = [a\ b]$, $V_1^{21} = c$, $Z_1^1 = z_1^{11} = a$, and $Z_1^2 = z_1^{21} = c$. The signs of κ_1^{11} and κ_1^{21} are determined by the signs of α and δ.

Part (iii) of Theorem 4 then states that two of the eigenvalues of the perturbed matrix

$$A(\epsilon) = (J - \epsilon D)S \tag{5.2}$$

satisfy, for $q = 1, 2$,

$$\mu_1^{11q} = \lambda_1 \pm \sqrt{\xi_1^{11}}\sqrt{\epsilon} + o(\sqrt{\epsilon}),$$

where

$$\xi_1^{11} = -\overline{\sigma_1^1}\kappa_1^{11} a^*SDSa = i\kappa_1^{11} a^*SDSa.$$

Part (iv) states that the third eigenvalue of the perturbed matrix corresponding to the unperturbed eigenvalue λ_1 satisfies

$$\mu_1^{211} = \lambda_1 + \xi_1^{21}\epsilon + o(\epsilon),$$

where

$$\xi_1^{21} = -\kappa_1^{21}\left(c^*SDSc - \frac{|a^*SDSb|^2}{a^*SDSa}\right).$$

Specifically, suppose that J has the form (1.2) and

$$S = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1/4 \\ 0 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & -1/4 & 0 & 0 \end{bmatrix}.$$

This matrix S has two negative and four positive eigenvalues. The matrix JS has one pair of eigenvalues $\lambda_1 = \pm 0.5i$, each with multiplicity three and two Jordan blocks. After carrying out the computations described above we find

$$a \approx \begin{bmatrix} 0.6233 - 0.3338i \\ 0 \\ -0.3117 + 0.1669i \\ 0.3338 + 0.6233i \\ 0 \\ 0.6677 + 1.2467i \end{bmatrix}, \quad b \approx \begin{bmatrix} 0 \\ 0 \\ -0.3338 - 0.6233i \\ 0 \\ 0 \\ -2.4934 + 1.3353i \end{bmatrix}, \quad c \approx \begin{bmatrix} 0 \\ 0.7071i \\ 0 \\ 0 \\ -1.414 \\ 0 \end{bmatrix}$$

with $\alpha = -i$, $\delta = 1$, $\kappa_1^{11} = -1$, $\kappa_1^{21} = 1$.

Arbitrarily choosing

$$D = E^T E, \quad E = \begin{bmatrix} 1 & 2 & 3 & -1 & -2 & -3 \\ 1 & -2 & 3 & 1 & 2 & -3 \end{bmatrix}$$

we have

$$\xi_1^{11} = -ia^* SDSa \approx -13.0625i$$

and

$$\xi_1^{21} = -(c^* SDSc - \frac{|a^* SDSb|^2}{a^* SDSa}) \approx -4.904306.$$

Therefore two of the eigenvalues of (5.2) satisfy, for $q = 1, 2$,

$$\mu_1^{11q} = 0.5i \pm (2.555631 - 2.555631i)\sqrt{\epsilon} + o(\sqrt{\epsilon}),$$

while the third satisfies

$$\mu_1^{211} = 0.5i - 4.904306\epsilon + o(\epsilon).$$

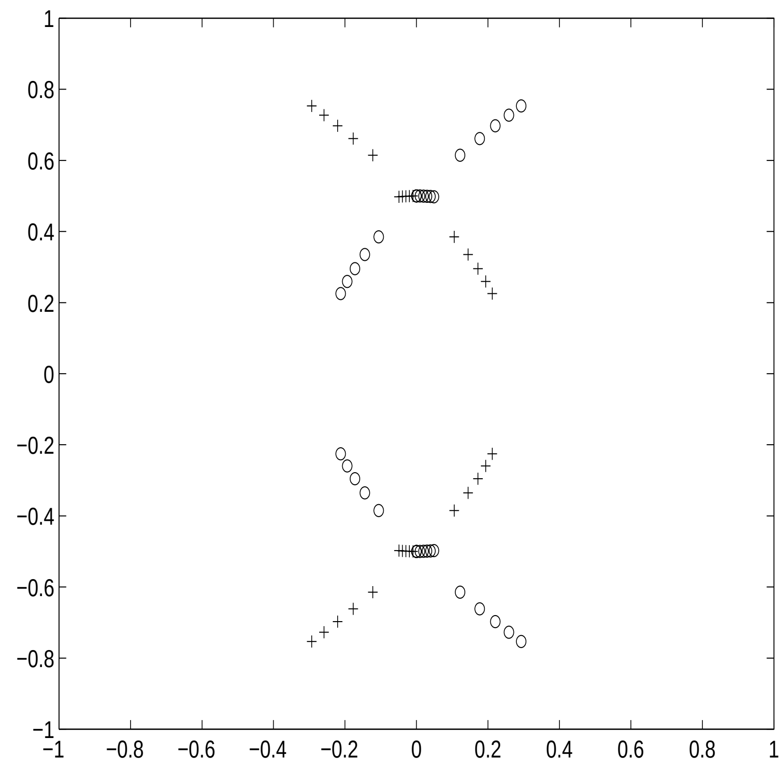

Figure 1: The splitting of an eigenvalue with algebraic multiplicity three and geometric multiplicity two. Plus signs (respectively circles) show eigenvalues of $(J - \epsilon D)S$ for equally spaced values of $\epsilon > 0$ (respectively $\epsilon < 0$).

These results can be compared with a numerical computation of the eigenvalues of (5.2) for $\epsilon = 10^{-6}$. Those with positive imaginary part are approximately

$$\begin{array}{c} -0.002559674 + 0.502555636i \\ 0.002551578 + 0.497444364i \\ -0.000004904306 + 0.5i, \end{array}$$

confirming the formulas derived above.

Figure 1 plots the eigenvalues of (5.2) for 11 discrete choices of ϵ uniformly spaced between -0.01 and 0.01, using the symbol "+" for positive values of ϵ and "o" for negative values of ϵ. Attention may be restricted to the upper half-plane. Notice that two of the eigenvalues move much more rapidly than the third, and that the "fast" eigenvalues change directions through an angle of $\pi/2$ as they pass through $0.5i$ ($\epsilon = 0$). By comparison the third eigenvalue moves very slowly and the symbols centered on the distinct perturbed values overlap in the plot. This sharp distinction between "strong" and "weak" interaction of eigenvalues is well known (e.g.[21]), and is a direct consequence of the Jordan structure of JS. The plot illustrates the main result of this paper, which is that the number of eigenvalues in the right half-plane for positive ϵ is precisely equal to the number of negative eigenvalues of S, in this case two.

6. Nonsymmetric Dissipative Perturbations

Theorems 1, 2, 3 and 4 all hold in the presence of more general dissipative perturbations in which D is nonsymmetric, but its symmetric part $D+D^T$ is positive semidefinite, with the analogue of hypothesis (2.2) holding, namely $S(D+D^T)S$ positive definite on the geometric eigenspaces for imaginary eigenvalues. In the homotopy proof of Theorem 1, D is merely broken into its symmetric and skew-symmetric parts. The proofs of Theorems 3 and 4 are unchanged. Theorem 2 is proven from Theorem 3 in the semisimple case by remarking that with the notation $T_j = Z_j^* SDSZ_j$, $T_j + T_j^*$ is positive definite, and consequently the matrix T_j is *inertia preserving* [3], i.e. the half-planes in which the eigenvalues of $K_j T_j$ lie are determined by the signs in K_j.

To prove Theorem 2 from Theorem 4 for nonsymmetric dissipative perturbations, one needs in addition only a generalization of the classic result for positive definiteness of Schur complements. If

$$M = \begin{bmatrix} F & B \\ G & C \end{bmatrix},$$

and $M + M^*$ is positive definite, then $T + T^*$ is also positive definite, where T is the Schur complement $T = C - GF^{-1}B$ [6], Theorem 4.1.5.

7. Nondissipative Perturbations

If $D(x)$ is skew-symmetric then the modified dynamics (1.6) remains within the class of Hamiltonian systems, but with a perturbed Poisson structure matrix $J(x)-\epsilon D(x)$. (Strictly the perturbation $D(x)$ should also be such that the Jacobi identity holds for $J(x)-\epsilon D(x)$, but we make no use of that property.) The perturbation formulas of Theorems 3 and 4 hold exactly as before, assuming, in the case of Theorem 4, that SDS is nonsingular on the geometric eigenspaces for imaginary eigenvalues. However, no analogues of Theorems 1 and 2 hold. Indeed, sharp counts on the number of eigenvalues in the right half-plane are not possible without detailed knowledge of the perturbation D. Nevertheless, Theorems 3 and 4 can be used to recover a variant of classic results characterizing Hamiltonian systems that are *strongly stable*. In this context, and anticipating connections with the classic theory that are described in the next section, we shall say that a Hamiltonian system linearized at a spectrally stable equilibrium point is strongly (or parametrically) stable, if the equilibria of all perturbed systems of the form (1.6) with $D=-D^T$ are spectrally stable.

Suppose therefore that D is skew-symmetric. Assume first that the imaginary eigenvalue λ_j is semisimple. From Theorem 3, the derivatives of the perturbed eigenvalues $\mu_j^q(\epsilon)$ are the eigenvalues of $-K_jT_j$, or equivalently T_jK_j, where $T_j = Z_j^*SDSZ_j$, and $K_j = Z_j^*SZ_j$ is a diagonal matrix of signs. Since T_j is skew-Hermitian and K_j is real diagonal, T_jK_j is Hamiltonian, which is not surprising, since the derivatives of the $\mu_j^q(\epsilon)$ must have two-fold symmetry. When the signs in K_j are all the same, i.e. S is definite on the eigenspace, it follows that the derivatives of the eigenvalues $\mu_j^q(\epsilon)$, i.e. the eigenvalues of T_jK_j, are all imaginary. If K_j contains mixed signs, it does *not* necessarily follow that an eigenvalue $\mu_j^q(\epsilon)$ moves off the imaginary axis. This indeterminacy in the effect of Hamiltonian perturbations is exactly analogous to the lack of sharpness in the correspondence between the eigenvalues of (1.4) and (1.5). However, it is clear that *some* skew-symmetric perturbation D moves eigenvalues off the imaginary axis whenever K_j has mixed signs. Therefore we say that λ_j is *not* strongly stable in this case.

Now, again assuming that D is skew-symmetric, suppose that the imaginary eigenvalue λ_j has at least one nontrivial Jordan block. Suppose for simplicity that there is only one such block, and that it has dimension two, with corresponding eigenvector z_j. Theorem 4 shows that this block is associated with two perturbed eigenvalues

$$\lambda_j \pm \sqrt{\xi_j}\sqrt{\epsilon} + o(\sqrt{\epsilon}),$$

where

$$\xi_j = i\kappa_j\tau_j, \quad \tau_j = z_j^*SDSz_j, \quad \kappa_j = \pm 1.$$

Since τ_j is imaginary, ξ_j is real, so the eigenvalues must split in opposite directions either along the imaginary axis or along the line passing through λ_j parallel to the real axis. It is impossible to know which case applies without knowing the sign of the imaginary quantity τ_j, but for some skew-symmetric D, the perturbation is unstable. If there is a Jordan block of size $m > 2$, the splitting must take place along rays separated by angles of $2\pi/m$, so eigenvalues move off the imaginary axis for all skew-symmetric D. Without detailed knowledge of the perturbation D it is not possible to count how many eigenvalues move into the right half-plane. But for all D, at least one eigenvalue moves into the right half-plane.

In summary, strongly stable systems are precisely those for which all eigenvalues λ_j are semisimple, with S positive definite on each eigenspace.

8. Relations with Classic Results

The conclusions of the last section are completely consistent with the Gel'fand-Krein-Lidskii-Moser strong stability theorem [8], [12], [14], [18] (p. 56), [19], [24] (p. 192). In fact, the remarks of Section 7 can be regarded as a paraphrase of the strong stability theorem, with the following three differences.

The first difference is that the classic form of the strong stability theorem actually pertains to linearization about *periodic* solutions, so that a linear system with periodic coefficients arises, and the result is accordingly usually described in terms of the movement of the *Floquet multipliers*, which are eigenvalues of the symplectic monodromy matrix. Since Hamiltonian and symplectic matrices are related by matrix exponentiation, an eigenvalue of a Hamiltonian matrix in the right half-plane corresponds to an eigenvalue of the associated symplectic matrix outside the unit circle. Consequently, when specialized to equilibrium points, the strong stability theorem provides information concerning eigenvalue movement with respect to the imaginary axis (Moser, [19], especially p. 112). Conversely, our results can be couched in terms of symplectic matrices. But in that context, the motivation for the physically appropriate form of dissipative perturbation is less clear.

Second, the original motivation for the strong stability theorem was to characterize those linearly stable systems that remain stable under Hamiltonian perturbations. It is therefore natural to restrict attention to the semisimple case, because unperturbed eigenvalues with nontrivial Jordan blocks correspond to unstable modes of the linearized dynamics with polynomial growth in time. However, in order to obtain the results of primary interest here, namely the sharp characterization of exponentially growing modes in the presence of dissipation, it is necessary to consider the more intricate case of perturbation of general Jordan structures, as described by Theorem 4.

Third, the family of perturbations that is considered in the classic form of the strong stability theorem is

$$B(\epsilon) = J(S + \epsilon E), \tag{8.1}$$

which stands in contrast to (2.1). In (8.1) the perturbation is Hamiltonian whenever E is symmetric, which is naturally associated with perturbations in the Hamiltonian $H(x)$. In (2.1) the perturbation is Hamiltonian whenever D is skew-symmetric, which is naturally associated with perturbations in the Poisson structure matrix $J(x)$. Since J and S are both assumed to be nonsingular, arbitrary linear perturbations of JS can be factored into either the form (2.1) or the form (8.1). However Darboux's Theorem and associated changes of coordinates must be invoked to investigate the full connections between the two schemes. For our purposes, where the primary focus is on dissipative perturbations, the family (2.1) is more natural because of the simple relationship with the damped nonlinear system (1.6).

Nonetheless, it is of interest to briefly consider extension of Theorems 3 and 4 to the perturbation family (8.1). Instead of (3.3), we have

$$\frac{d}{d\epsilon}B(0) = JE.$$

Let us restrict attention to the semisimple case. Note that with J of the form (1.2), $SZ_j = -\lambda_j JZ_j$, so that K_j can also be written in the form $-\lambda_j Z_j^* JZ_j$. A related observation is that, again assuming (1.2), the left eigenvector basis $Y_j = K_j Z_j^* S$ can also be written $Y_j = -\lambda_j K_j Z_j^* J$. Consequently the derivatives of the eigenvalues of B corresponding to λ_j are the eigenvalues of

$$Y_j JEZ_j = -\lambda_j K_j Z_j^* J^2 EZ_j = \lambda_j K_j T_j \tag{8.2}$$

where $T_j = Z_j^* EZ_j$. Suppose E is symmetric, so that (8.1) is a Hamiltonian perturbation. If we further assume that either S or E is definite on the eigenspace, so that either K_j or T_j has a Hermitian square root, it follows that the eigenvalues of $\lambda_j K_j T_j$, i.e. the derivatives of the eigenvalues of B, are imaginary. If neither S nor E is definite on the eigenspace, the derivatives may be non-imaginary.

These conclusions comprise the classic form of the strong stability theorem, translated from symplectic to Hamiltonian matrices, and restricted to the semisimple case. (In the non-semisimple case strong stability is not possible, as already explained in the previous section.) The signs in the matrix K_j are equivalent to the *Krein signatures* of the positive imaginary eigenvalue λ_j, as defined in [7], p. 11, for example. In the classic discussions, Krein signatures are defined such that λ_j and its conjugate $-\lambda_j$ have signatures of opposite signs. In that case, there are always a total of n positive and n negative Krein signatures.

It is perhaps of interest to remark that, according to the discussion in [8], pp. 147–149, in his original treatment Krein [12] considered perturbations of the form (8.1) for two choices of perturbation E. First he took E to have the skew-Hermitian form $E = iF$, where F is real, symmetric and positive definite, and used an argument, directly analogous to the proof of Theorem 1 given here, to show that no eigenvalue remains on the imaginary axis. The symmetry of time reversal was exploited to claim that n eigenvalues move into each half-plane. This dichotomy was used to label each eigenvalue as being of either Type 1 or Type 2. Then for the second class of perturbations, namely $E = F$, a real, symmetric and positive definite perturbation, the classification was combined with a homotopy argument to conclude that eigenvalues cannot move off the imaginary axis to form complex quadruplets unless a Type 1 and Type 2 eigenvalue coincide. These conclusions also follow immediately from (8.2). With $E = iF$, T_j is i times a Hermitian, positive-definite matrix, so the derivatives of the pure imaginary eigenvalues λ_j are all real. The signs of any one group of perturbed eigenvalues are determined by the inertia of S restricted to the corresponding unperturbed eigenspace. In total, n eigenvalues move left and n move right, because the derivatives are real with a factor of λ_j, so the sign pattern for the perturbation of λ_j is the opposite to that for the conjugate eigenvalue $-\lambda_j$. Type 1 and 2 eigenvalues coincide precisely when S is indefinite on a given eigenspace, and, as already explained, that is the case when Hamiltonian perturbations can move eigenvalues off the imaginary axis.

In recent work, MacKay [15] analyzes dissipative perturbations using the perturbation family (8.1) for general matrices E. He gives a formula closely related to (8.2) in the case where λ_j is simple, and interprets it in terms of energy and energy decay rate of the linearized dynamics. MacKay remarks that extension to the case of multiple eigenvalues would be of interest. This generalization is provided by our results.

9. Lagrangian Dynamics with Rayleigh Dissipation

In this section we shall demonstrate that a large class of systems arising in classical mechanics are encompassed by the framework that was adopted in Section 1. Consider autonomous Lagrangian systems, i.e. second-order equations of the form

$$-\frac{d}{dt}L_{\dot q} + L_q = 0. \tag{9.1}$$

Here the Lagrangian $L(q, \dot q) : \Re^{2n} \to \Re$ is a given function of the generalized coordinates $q(t) \in \Re^n$ and associated generalized velocities $\dot q(t)$, and the subscripts denote partial derivatives. Provided that the matrix $L_{\dot q \dot q}$ is nonsingular, the definition

$$p \equiv L_{\dot q}(q, \dot q)$$

of the conjugate momenta $p \in \Re^n$ can be inverted locally, yielding

$$\dot q = Q(q, p).$$

As is well known, the second-order system (9.1) is equivalent to a first-order Hamiltonian system of the form (1.1) with the canonical structure matrix (1.2), $x = (q, p)$, and

$$H(q, p) = pQ(q, p) - L(q, Q(q, p)). \tag{9.2}$$

The Lagrangian is often a convex function of the generalized velocities, for example a positive definite quadratic form, in which case, with appropriate growth conditions, the Hamiltonian (9.2) can be defined globally by the standard Legendre transform of convex analysis. We shall not make this convexity assumption, although we shall exploit the fact that $H_{pp} = L_{\dot q\dot q}^{-1}$, so that nonsingularity of $L_{\dot q\dot q}$ implies invertibility of H_{pp}.

The standard way to introduce dissipation into Lagrangian dynamics uses a Rayleigh dissipation function of the form $\frac{1}{2}\dot q^T R(q)\dot q$, with $R(q)$ a symmetric positive definite $n \times n$ matrix ([20], Chpt. X). Then the conservative dynamics (9.1) is modified to the dissipative dynamics

$$-\frac{d}{dt}L_{\dot q} + L_q = \epsilon R(q)\dot q, \tag{9.3}$$

where ϵ is a positive scaling factor. If the momenta p are introduced as before, we find that the dissipative second-order dynamics (9.3) is equivalent to the perturbed Hamiltonian system (1.6) where x, J, and H are defined as above, and the positive semidefinite, symmetric perturbation matrix $D(x)$ is defined by

$$D(q,p) = \begin{bmatrix} 0 & 0 \\ 0 & R(q) \end{bmatrix}. \tag{9.4}$$

We are interested in the dynamics linearized about equilibrium solutions $(q(t), \dot q(t)) \equiv (q_e, 0)$. Such solutions are equilibria of (9.1) and (9.3) precisely when $L_q(q_e, 0) = 0$. At such points the linearization of (9.3) is

$$-\hat L_{\dot q\dot q}\ddot u + \left\{\hat L_{\dot q q}^T - \hat L_{\dot q q}\right\}\dot u + \hat L_{qq}u = \epsilon \hat R \dot u, \tag{9.5}$$

where $u(t)$ is the linearized variable, and $\hat L_{\dot q\dot q} = L_{\dot q\dot q}(q_e, 0)$, etc. Of course (9.5) is itself a Lagrangian system, with a quadratic Lagrangian, that is perturbed by Rayleigh dissipation. The associated dissipatively perturbed, linear Hamiltonian system can either be found by the usual transformation from (9.5), or from linearization of (1.6) about (q_e, p_e). The associated quadratic Hamiltonian is $\frac{1}{2}(u,w)^T S(u,w)$ where

$$S \equiv \begin{bmatrix} H_{qq} & H_{qp} \\ H_{pq} & H_{pp} \end{bmatrix} = \begin{bmatrix} \hat L_{q\dot q}\hat L_{\dot q\dot q}^{-1}\hat L_{\dot q q} - \hat L_{qq} & -\hat L_{q\dot q}\hat L_{\dot q\dot q}^{-1} \\ -\hat L_{\dot q\dot q}^{-1}\hat L_{\dot q q} & \hat L_{\dot q\dot q}^{-1} \end{bmatrix}, \tag{9.6}$$

and (u, w) are the linearized variables associated with (q, p).

Lemma 2. *Suppose that S is an invertible matrix of the form (9.6) with the H_{pp} block also invertible, D is a matrix of the form (9.4) with $\hat R$ positive definite, and J is the canonical matrix (1.2). Then $z^*SDSz > 0$ for all eigenvectors z of JS, i.e. hypothesis (2.2) is automatically satisfied.*

Proof: Suppose not. Then there exists an eigenvector $z = (z_1, z_2)$ with $Sz = (z_3, 0)$, i.e.

$$\begin{bmatrix} H_{qq} & H_{qp} \\ H_{pq} & H_{pp} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_3 \\ 0 \end{bmatrix}, \tag{9.7}$$

and

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} z_3 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

But by hypothesis $\lambda = 0$ is not an eigenvalue, so $z_1 = 0$. Then from the second equation in (9.7) and the invertibility of H_{pp}, we conclude that $z_2 = 0$ also, a contradiction. Q.E.D.

Lemma 2 allows application of Theorems 1 and 2 to this class of dissipatively perturbed Lagrangian systems with no explicit knowledge of any eigenvector required. It is by no means the case that all Hamiltonian systems are of the form described in this section, but many are. Indeed considerable work has been expended on such systems, and on further special cases of such systems. For example, many Lagrangians arising in classical mechanics are decoupled with kinetic and potential energies of the form

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T T(q)\dot{q} - V(q), \tag{9.8}$$

and associated Hamiltonian

$$H(q, p) = \frac{1}{2}p^T T^{-1}(q)p + V(q), \tag{9.9}$$

where $T(q)$ is a symmetric positive definite matrix defining a kinetic energy that is pure quadratic in the velocities, and $V(q)$ is the potential. In this special context the off-diagonal blocks $\hat{L}_{q\dot{q}}$ and H_{pq} vanish, and the issues discussed in this paper are moot, for it is straightforward to show that there is a sharp correspondence between the number of negative eigenvalues of S and the number of eigenvalues of JS in the right half-plane, even in the absence of any dissipative perturbation. (There are delicate questions concerning characterization of stability properties of *degenerate* equilibria in such systems, e.g.[1], p. 271, but they are of an essentially different nature.) One place to find a review of such classic issues is [16].

Accordingly it can be seen that the crucial terms are the off-diagonal entries in the Hamiltonian (9.6), or equivalently the terms in the linearized Lagrangian dynamics (9.5) involving the velocities $\dot{u}$ with skew-symmetric coefficients. Since the time of Kelvin such terms have been called *gyroscopic*, because they are often associated with effects of rotation, and, in particular, the stability of steady spins of gyroscopes. The lengthy discussions in [23], Article 345 and [5], Chpt. V consider general perturbations of equations of the form (9.5) with $\hat{L}_{\dot{q}\dot{q}}$ positive definite. See also [24], p. 333, where derivative formulas are given for eigenvalues arising from modes of (9.5) when subject to general perturbations, but only for the case of semisimple eigenvalues.

The inertia theorem for Schur complements can be applied to matrices S of the specific form (9.6) to conclude that the number of negative eigenvalues of the $2n \times 2n$ matrix S equals the sum of the numbers of negative eigenvalues of the two $n \times n$ blocks $\hat{L}_{\dot{q}\dot{q}}$ and $-\hat{L}_{qq}$. For Lagrangians of the form

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T T(q)\dot{q} + Q(q)^T \dot{q} - V(q), \tag{9.10}$$

that are quadratic in the velocities $\dot{q}$ with $T > 0$, we may further conclude that at equilibria, $\hat{L}_{\dot{q}\dot{q}} = \hat{T}$ has no negative eigenvalue, and $-\hat{L}_{qq} = V_{qq}(\hat{q})$. Thus for Lagrangian systems of the form (9.10), Theorems 1 and 2 can be applied with the number of negative eigenvalues of S being replaced with the number of negative eigenvalues of the Hessian $\hat{V}_{qq}$ of the potential. (Moreover Lemma 2 implies that hypothesis (2.2) is automatically satisfied.) In particular it may be concluded that in the presence of complete Rayleigh dissipation, the only (nondegenerate) equilibria that are stable are minima of the potential, a result sometimes known as the Kelvin-Tait-Chetayev Theorem [10], Chpt. 5.10.

Rayleigh dissipation is also discussed in [4]. While the primary focus is on the case of relative equilibria, they do consider the case of equilibria. Their attention is restricted to the important, but nevertheless special, case of Hamiltonians of the form (9.9). They derive perturbation formulas, but only for the case of simple eigenvalues. They also adopt a Lyapunov-type approach to prove that at critical points that are not minima there is at least one unstable mode of the linearized damped dynamics. Our results give a sharp count on the number of unstable modes for a more general class of Hamiltonians.

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