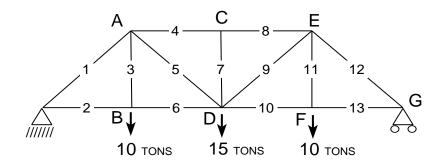
A Truss Problem

A typical task in structural engineering is to design a bridge to be strong enough to withstand a certain load. Consider the following *plane truss*, which is a set of metal bars connected by frictionless pin joints. The symbol at the left end of the truss indicates that it is fixed at that end, while the symbol at the right end indicates that the truss is free to move horizontally, but not vertically. The three arrows pointing down represent loads on the truss. These loads are 10 tons, 15 tons, and 10 tons respectively. The same picture appears in Ex 2.3 of the text, although the notation and loads are slightly different.



The problem is to solve a certain linear system of equations for the *internal* forces in the bars. A positive internal force indicates that the bar is being extended (pulled apart a little), by the load, while a negative internal force indicates that the bar is being compressed. It is assumed that, as long as the internal forces are not too big, bars will not be stretched or compressed more than a tiny amount: thus the structure does not collapse, but remains in equilibrium. By computing the internal forces, an engineer has more information as to whether the truss is indeed strong enough to withstand the load.

There are two linear equations for each internal joint in the truss, representing forces in the horizontal and vertical direction which must balance at the joints. Let us denote the internal forces by x_1, x_2, \ldots, x_{13} , corresponding to the numbers on the bars in the illustration. The balancing of forces at joint C in the *horizontal* direction gives the equation

 $x_4 = x_8$

while the balancing of forces at joint C in the vertical direction gives simply

 $x_7 = 0.$

The balancing of forces at joint B in the *horizontal* direction gives

 $x_2 = x_6$

while the *vertical* direction at joint B gives

$$x_3 = 10.$$

The "10" comes from the 10 ton vertical load at joint B. The balancing of forces at joint A is a little more complicated, since it involves two bars oriented at an angle of 45 degrees as well as a horizontal and a vertical bar. Let $\alpha = \cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$. Then the balancing of *horizontal* forces at joint A gives the equation

$$\alpha x_1 = x_4 + \alpha x_5$$

and the balancing of *vertical* forces at joint A gives

$$\alpha x_1 + x_3 + \alpha x_5 = 0.$$

There are also horizontal and vertical force equations at joints D, E and F which can be derived using the same ideas. These amount to 12 equations altogether. The 13th equation comes from the right end point G: since this end point is free to move horizontally, but not vertically, there is just one force equation, balancing the forces horizontally:

$$x_{13} + \alpha x_{12} = 0.$$

Thus, we have a total of 13 linear equations defining the 13 internal force variables. These equations are written out in detail in the text. Set up this system of equations and solve it any way you like; this is very easy to do in Matlab using $x=A\b$.

Then generalize this set of equations by writing a function that sets up the system of equations for a variable-sized truss, with k sections exactly like the section ABCDEF instead of one, where k is an input parameter to the function. The function should return the matrix A and vector b defining the system of equations for a variable-sized truss. This will require some careful thought. Start by sketching the larger truss on paper and carefully writing down the relevant equations systematically; working together with a classmate for this part may be quite helpful and is permitted as long as it is documented that you worked together. Make sure you number the variables in the appropriate order, so that you recover your original answer when k = 1. The load is a vector whose length depends on k (what is it?) and should be provided to the function as a second input parameter. Make sure you include plenty of comments explaining the code. Test your function, solving the system of equations that it sets up for small k (including k = 1) and carefully look at the output to see if it makes sense before going on to the next part.

- 1. Solving the system: Solve the system using a load vector that increases regularly like this from left to right: $10, 20, 30, \ldots$ Print the computed internal force vector x for k = 10 (as numbers, you don't need to plot these values).
- 2. Sparsity: Graphically display the positions of the nonzero entries in the matrix A for k = 10; this is easy in Matlab using spy. Do the same for A⁻¹ (the *inverse* of A), and for the L and U factors obtained, in Matlab, by inv(A) and [L,U,P] = lu(A) respectively. By looking at the permutation matrix P you can see whether pivoting occurs in the LU factorization; does it? How sparse are A⁻¹, L and U, compared with the sparsity of A?
- 3. **Timing Comparsion:** Experiment with how long it takes to solve the system of equations for k that is large enough that timing comparisons are meaningful (in Matlab you can use cputime). Compare the following:
 - direct solve of Ax = b without any sparse matrix techniques (in Matlab, by x=A\b).
 - getting x by first computing A^{-1} and multiplying it onto b
 - getting x by first computing the L and U factors and permutation P and then solving two triangular systems (you can use \ for this; the triangular structure is discovered and exploited). Don't forget to apply the permutation to b first, unless it is built into L (this is done, for example, if you use [L,U] = lu(A) in Matlab). (This is exactly what is going on behind the scenes when you type x=A\b.)
 - the same 3 again, but with your function modified so that A is stored using a *sparse* data structure where the zeros are not explicitly stored (very easy in Matlab: type help sparse). This allows handling much larger k than is otherwise possible.

Can you conclude anything about the running times as a function of k: are they something like O(k), $O(k^2)$, $O(k^3)$ or what? Summarize your results concisely, ideally using graphical output; do *not* turn in pages of supporting output.